He who would measure the Earth must
first measure the Stars.

THE ELEMENTS

OF

GEODETIC ASTRONOMY

FOR CIVIL ENGINEERS

BY

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PREFACE

This book was written for the students in the course in civil engineering at the Ohio State University and to put in permanent form the author's manner of presenting this subject. After an experience of over sixteen years in teaching geodetic astronomy to engineering students, he is firmly convinced of the value of this subject as a training for the young engineer. Practical astronomy gives, as nothing else does, experience in precise measurement and in handling long and intricate calculations.

The equipment which this book presupposes is not expensive. An investment of $4000 would be sufficient to provide amply for a class of twenty.

In the derivation of formulæ, the author has sought directness of method rather than elegance of mathematical analysis. In the numerical examples, taken from observations made at the Emerson McMillin Observatory, he has aimed to furnish the student with models of correct forms for note-book record as well as of correct forms for reduction sheets. He has placed much emphasis upon the theodolite, as that is the instrument of the engineer and the one above all others with which he should become familiar.

Though, in a work of this kind, it is impossible to state all sources which have been consulted, the author

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wishes to mention the classics of Chauvenet and Doolittle, and Professor Perry's gem of popular exposition "Spinning Tops". He also takes this occasion to express his obligation to his assistant, Mr. B. F. Maag, who has not only gone over all the proof and checked all the calculations and formulæ, but has also made many valuable suggestions during the progress of the work.

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GEODETIC ASTRONOMY

CHAPTER I.

FUNDAMENTAL CO-ORDINATES

The sky on a cloudless day gives us the impression of an immense hollow sphere; at night the stars seem like points fixed to its inner surface. This sphere seems to travel with us and wherever we go we seem to be at its center. The distances of the stars are so great that for most purposes we may consider them infinite. We shall, therefore, define the celestial sphere as a sphere whose center is the eye of the observer and whose radius is infinite. When we speak of the position of a star, a planet, the sun, or moon, we shall mean the position of its projection upon the inner surface of the celestial sphere.

To fix the position of a star on the surface of the celestial sphere, we must have a great circle formed by a fundamental plane of reference and a fixed or zero point in this great circle. Two arcs will then define the star's position. One arc gives the distance of the star from the great circle; the other, the distance along the great circle from the zero point to the foot of a second great circle passing through the star and perpendicular to the first great circle.

These two arcs are called the star's co-ordinates, and obviously there will be as many systems of co-ordinates as there are fundamental planes of reference.

Any plane is completely defined when we know a

point through which it passes, and a line to which it is perpendicular. In each and every system the center of the celestial sphere, i. e., the eye of the observer, is taken as the fixed point. Two lines naturally suggest themselves as convenient lines of reference: first, the plumb line; second, the earth's axis. Corresponding to these lines we have two co-ordinate systems. In the first, the fundamental plane is perpendicular to the plumb line; in the second, to the earth's axis. We may then define the plane of the horizon as a plane passing through the eye of the observer, perpendicular to the plumb line; and the plane of the equator as a plane passing through the eye of the observer, perpendicular to the earth's axis. The corresponding great circles in which these planes cut the celestial sphere are called the horizon and the equator, respectively.

There is a third system of co-ordinates used in certain branches of astronomy. In this the fundamental plane is fixed without reference to any line. The earth moves around the sun in an ellipse called the earth's orbit. A plane passing through the eye of the observer, parallel to the plane of this orbit, is taken as the fundamental plane in the third system and is called the plane of the ecliptic. The corresponding great circle is called the ecliptic.

The plane of the ecliptic passes very near both the center of the earth and that of the sun; hence the sun, as seen from the earth, will always appear as projected upon the ecliptic and, on account of the earth's orbital motion, will apparently travel completely around it in the course of a year. The stars, being more than 250 000 times as far off as the sun, will not have their positions appreciably changed by the earth's orbital motion. The sun will appear to make, therefore, an annual journey of 360 degrees

through the stars, traveling always along the ecliptic. We may give, therefore, a second definition of the ecliptic as follows: the ecliptic is the apparent annual path of the sun through the stars.

We are now ready to define for each system the first arc necessary to fix the position of an object, or rather of its projection on the celestial sphere. Using the term distance to mean arc, or angular distance, i. e., degrees, minutes, and seconds, we may define a star's distance from the horizon as its altitude (h), its distance from the equator as its declination (δ), and its distance from the ecliptic as its latitude (B).

We have seen above that a second great circle, perpendicular to the fundamental plane, is needed to define the second co-ordinate. These circles are called vertical circles in the first system, hour circles in the second, and secondaries to the ecliptic in the third. A vertical circle is therefore a great circle formed by a plane passing through the plumb line. An hour circle is a great circle formed by a plane passing through the earth's axis. That particular plane which passes through both the earth's axis and the plumb line is called the plane of the meridian. The corresponding great circle is called the meridian. The meridian may be defined in two ways: as the vertical circle which passes through the earth's axis, or as the hour circle which passes through the plumb line.

The two points in which the earth's axis cuts the celestial sphere are called the north and south poles. The plumb line cuts the celestial sphere in two points: the one, directly over head, is called the zenith; the other, directly under foot, is called the nadir.

We are now ready to define the second arc in the first two systems. In the horizon system, the azimuth, (A), is

the distance along the horizon from the meridian to the foot of the vertical circle through the star. It is customary to measure this from the south around through the west. In the equator system, the hour angle, (t), is the distance along the equator from the meridian to the foot of an hour circle through the star. Hour angles to the east are negative; to the west, positive. Both of these arcs may also be defined as angles between two great circles. The azimuth is the angle at the zenith, between the meridian and the vertical circle through the star. The hour angle is the angle at the pole, between the meridian and the hour circle through the star. Both methods of regarding these co-ordinates are of equal importance.

To fix the zero point in the ecliptic system we make use of the equator. This plane is inclined to the plane of the ecliptic by about 23½°. The line of intersection of these two planes is called the line of equinoxes, and the points in which this line cuts the celestial sphere are called the equinoxes. That equinox at which the sun passes from the south to the north side of the equator is called the vernal equinox, the other is called the autumnal equinox. The sun is in the vernal equinox about the 21st of March, and in the autumnal equinox about the 21st of September of each year.

As the earth moves in its orbit about the sun its axis remains constantly parallel to itself. This causes the line of equinoxes to maintain always the same direction and its intersections with the celestial sphere to be fixed points among the stars. The vernal equinox is taken as the zero point in the ecliptic system. A star's distance from the vernal equinox measured along the ecliptic to the foot of a secondary to the ecliptic passing through the star, is called its longitude.

There is a fourth system in frequent use. This is the same as the equator system except that the star's hour angle is replaced by its right ascension. The right ascension, (a), is the distance from the vernal equinox measured along the equator to the foot of the hour circle through the star. Right ascensions are positive when measured from the west toward the east.

Two more arcs are in frequent use; they are the star's zenith distance (z), and its north polar distance (P). As the names imply, the zenith distance is the distance from the zenith along a vertical circle to the star; the north polar distance is the distance from the north pole along an hour circle to the star. Since the distances from the zenith to the horizon, and from the pole to the equator, are each 90 degrees, we have the relations $z = 90° - h$ and $P = 90° - \delta$.

Fig. I, page 6, shows these co-ordinates as they would appear looking down on the plane of the horizon from the south-west at an elevation of 45 degrees. A tabular statement of all of the above definitions is given on page 7.

Several quantities must now be defined which refer to the position the observer occupies on the surface of the earth. They are his longitude and latitude. The longitude of any point on the earth's surface is the distance measured along the equator from some prime meridian, such, for example, as that of Washington or Greenwich, to the meridian passing through the point. There are three kinds of latitude used in astronomy. These are each defined as follows:—

1. Astronomical latitude, (ϕ), is the angle the plumb line makes with the plane of the equator. Obviously it is the distance along the meridian from the equator to the zenith, in other words the declination of the zenith.

SYSTEMS OF CO-ORDINATES

INDEX TO FIG. I.

O is the observer.

P is the pole. OP is the earth's axis.

Z is the zenith. OZ is the plumb line.

M, W, N, and E are the south, west, north, and east points of the horizon.

S is the star.

VO is the line of equinoxes.

TS = the altitude (h).

ZS = the zenith distance (z).

BS = the declination (δ).

FS = the star's latitude (B).

MAT = MZT = the azimuth (A).

CB = CPB = the hour angle (t).

VB = the right ascension (α).

VF = the star's longitude (L).

COZ = CZ = the observer's latitude (ϕ).

FIG. I.

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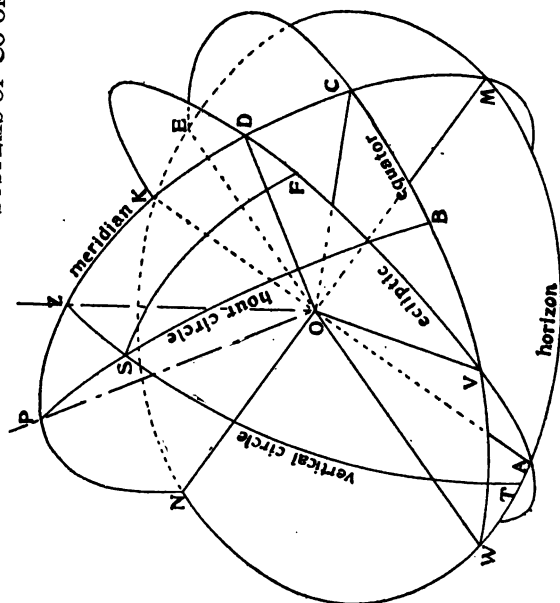
The plane of the $\begin{Bmatrix} \text{Horizon} \\ \text{Equator} \\ \text{Ecliptic} \end{Bmatrix}$ is a plane $\begin{Bmatrix} \text{perpendicular to the plumb line.} \\ \text{perpendicular to the earth's axis.} \\ \text{parallel to the plane of the earth's orbit.} \end{Bmatrix}$

A star's $\begin{Bmatrix} \text{Altitude . . .} \\ \text{Zenith Distance} \\ \text{Declination . .} \\ \text{Latitude . . .} \end{Bmatrix}$ is its distance from the $\begin{Bmatrix} \text{Horizon.} \\ \text{Zenith.} \\ \text{Equator.} \\ \text{Ecliptic.} \end{Bmatrix}$

A star's $\begin{Bmatrix} \text{Azimuth . . .} \\ \text{Hour Angle . .} \\ \text{Right Ascension} \\ \text{Longitude . .} \end{Bmatrix}$ is the distance along the $\begin{Bmatrix} \text{Horizon} \\ \text{Equator} \\ \text{Equator} \\ \text{Ecliptic} \end{Bmatrix}$ from the $\begin{Bmatrix} \text{Meridian . . .} \\ \text{Meridian . . .} \\ \text{Vernal Equinox} \\ \text{Vernal Equinox} \end{Bmatrix}$

to the foot of the $\begin{Bmatrix} \text{Vertical Circle . .} \\ \text{Hour Circle . . .} \\ \text{Hour Circle . .} \\ \text{Secondary to the Ecliptic} \end{Bmatrix}$ passing through the star.

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2. Geocentric latitude, (ϕ'), is the angle the line joining the given point with the center of the earth makes with the plane of the equator.

3. Geographical latitude, (ϕ), is the angle the normal to the earth's surface makes with the plane of the equator.

Geocentric differs from geographical latitude because the earth is an ellipsoid formed by rotating an ellipse about its minor axis, its major axis generating the plane of the equator. The normal at any point on such an ellipsoid will not, in general, pass through its center, but will intersect the equatorial plane between the center and the surface; hence, the normal will make a greater angle with this plane than the line which joins the given point with the center.

Astronomical latitude differs from geographical latitude because the earth's surface is not smooth nor is its crust of uniform density; and local irregularities, such as mountains above the surface or large deposits of denser material below, cause the plumb line to be deflected from its natural direction, namely, that of the normal to the earth's surface. The difference between these two is small and in most cases may be entirely neglected. Astronomical latitude, as its name implies, is that given directly by astronomical observations.

Geocentric latitude may be computed from the geographical latitude by the following formulæ:—

$$R \cos \phi' = F \cos \phi \text{ and } R \sin\phi' = \frac{\sin \phi}{G}$$

where R is the radius of the earth for the latitude ϕ in terms of its radius at the equator. The logarithms of F and G are given for the argument ϕ in the back part of the American Ephemeris under the heading '' Use of the Tables''.

CHAPTER II.

TRANSFORMATION OF CO-ORDINATES

We have seen in the last chapter that there are four systems of co-ordinates used to define a star's position. It frequently happens that we may observe two co-ordinates in one system, while we desire to know them in some other; or that we may observe one co-ordinate of one system, one of another, while we desire to know the two remaining co-ordinates of both systems. Two examples will help to make this clear.

Suppose at a known instant of time we measure the horizontal angle between a fixed mark and the sun. From the known time, by methods to be given later, we can compute the sun's hour angle, and from the American Ephemeris find its declination. This gives us the sun's co-ordinates in the equator system. If we transform these to the horizon system we obtain the sun's altitude and azimuth. Adding or subtracting, as the case may be, the measured angle between the mark and the sun, we obtain the azimuth of the mark, from which we can turn off a true north and south line.

As a second example, suppose we measure the altitude of a star at a given instant of time by our watch. From the American Ephemeris we can find the star's declination. We then have one co-ordinate in the horizon system and one in the equator system. Transforming these we

can find the star's hour angle and azimuth. From the hour angle we can find the true time at the instant we made the observation; comparing this with the observed time by our watch, we obtain its error on correct time.

The easiest way of making such transformations is by means of the astronomical triangle. A glance at Fig. I shows us that the pole, zenith, and star are at the vertices of a spherical triangle, of which the sides are P S = 90° — δ, Z S = 90° — h = z, and Z P = 90° — φ; and the angles are 180° — A at Z, t at P, and the so-called parallactic angle at S. It is a well known proposition of spherical trigonometry that if any three parts of a spherical triangle are given the remaining three may be computed. The two given co-ordinates, together with the observer's known latitude, give us the known parts.

Before proceeding to the solution of this triangle, it will be well to collect certain formulæ of spherical trigonometry. Let A B C be a spherical triangle, in which A, B, and C are the angles, and a is the side opposite A, b is that opposite B, and c that opposite C. We have then the following formulæ:—

(1) $\dfrac{\sin A}{\sin a} = \dfrac{\sin B}{\sin b} = \dfrac{\sin C}{\sin c}$

(2) $\cos a = \cos b \cos c + \sin b \sin c \cos A$

(3) $\operatorname{tg} \tfrac{1}{2} A = \sqrt{\dfrac{\sin(s - b)\,\sin(s - c)}{\sin s \,\sin(s - a)}}$, where

(4) $s = \tfrac{1}{2}(a + b + c)$

These four formulæ, from their symmetry, are very easy to remember and should be learned so as to be at the fingers' ends. The fifth lacks this quality of symmetry, hence it is difficult to remember; its derivation from (2) is easy and should be so learned that the student can write it out at once. To derive it we proceed as follows:— Ap-

plying (2) to the sides a and b we obtain

(6) $\cos a = \cos b \cos c + \sin b \sin c \cos A$

(7) $\cos b = \cos a \cos c + \sin a \sin c \cos B$

Substituting the value of cos b from (7) in (6),

$\cos a = (\cos a \cos c + \sin a \sin c \cos B)\cos c +$
$$\sin b \sin c \cos A$$

Expand and transpose the first term to the left-hand side of the equation, and we have

(8) $\cos a - \cos a \cos^2 c = (1 - \cos^2 c)\cos a =$
$$\sin a \sin c \cos c \cos B + \sin b \sin c \cos A$$

But $1 - \cos^2 c = \sin^2 c$. Substitute this value in (8) and divide by sin c, and we have

(5) $\cos a \sin c = \sin a \cos c \cos B + \sin b \cos A$

We will now proceed to the application of these formulæ to the solution of certain problems in the transformation of co-ordinates.

GIVEN:— The Altitude and Azimuth.

· REQUIRED:— The Declination and Hour Angle.

In the triangle P Z S, we have

$PZ = 90° - \phi$ $PSZ = q$

$PS = 90° - \delta$ $PZS = 180° - A$

$ZS = 90° - h$ $ZPS = t$

Formula (1) gives

$$\frac{\sin t}{\sin(90° - h)} = \frac{\sin(180° - A)}{\sin(90° - \delta)}$$

From this we obtain

(9) $\sin t \cos \delta = \sin A \cos h$

Formula (2) gives, if we call P S the side a,

$\cos(90° - \delta) = \cos(90° - h) \cos(90° - \phi) +$
$$\sin(90° - h) \sin(90° - \phi) \cos(180° - A)$$

which reduces to

(10) $\sin \delta = \sin h \sin \phi - \cos h \cos \phi \cos A$

Before we can apply (5) to this triangle we must

determine what letters of (5) are to represent the unknown parts, P S and Z P S. The following general rule applies to all cases where this formulæ is to be used in the transformation of co-ordinates. The last term of (5) contains a side and an angle not found in any other term, while the first and second terms contain two sides common to both. Always represent the unknown parts by the side and angle of the last term. Thus P S = b and Z P S = A. This fixes the remaining parts as follows: Z S = a, P Z = c, and P Z S = B. We then have

$$\cos(90° - h)\sin(90° - \phi) =$$
$$\sin(90° - h)\cos(90° - \phi)\cos(180° - A) +$$
$$\sin(90° - \delta)\cos t$$

This reduces to

(11) $\cos \delta \cos t = \sin h \cos \phi + \cos h \sin \phi \cos A$

These three formulæ, (9), (10), and (11), completely solve the problem, but they are not well adapted for the use of logarithms. In order to put them in a form convenient in practice we introduce two arbitrary constants, n and Z, defined by the equations,

(12) $n \cos Z = \sin h$
(13) $n \sin Z = \cos h \cos A$

We might equally as well have put $n \sin Z = \sin h$ and $n \cos Z = \cos h \cos A$, which would lead to a different but equally good set of formulæ. The student is warned against trying to remember this transformation in its exact form; the important thing is to make the right-hand sides of these equations, (12) and (13), functions of the star's co-ordinates.

Substitute (12) and (13) in (10) and (11) and we get

$$\sin \delta = n \cos Z \sin \phi - n \sin Z \cos \phi$$
$$\cos \delta \cos t = n \cos Z \cos \phi + n \sin Z \sin \phi$$

These reduce to

(14) $\sin \delta = n \sin(\phi - Z)$

(15) $\cos \delta \cos t = n \cos(\phi - Z)$

Divide (9) by (15), reduce, and we have

$$\text{tg } t = \frac{\sin A \cos h}{n \cos(\phi - Z)}$$

If we substitute in this the value of n, derived in (13), we obtain

(16) $\text{tg } t = \text{tg } A \dfrac{\sin Z}{\cos(\phi - Z)}$

Divide (14) by (15) and (13) by (12), simplify, and we have

(17) $\text{tg } \delta = \text{tg}(\phi - Z) \cos t$

(18) $\text{tg } Z = \cot h \cos A$

An additional formula may be derived as a check. Divide (13) by (15), reduce, and we have

(19) $\dfrac{\cos h \cos A}{\cos \delta \cos t} = \dfrac{\sin Z}{\cos(\phi - Z)}$

This check requires only that cos h and cos δ should be picked out of the tables, since the right-hand side of (19) has already been computed in (16).   Z is to be taken in any quadrant which satisfies the algebraic sign of tg Z. A complete example, together with remarks on methods of computing, is given at the end of the chapter.

There is a second case of frequent use in practice.

GIVEN:— The Declination and Altitude.

REQUIRED:— The Hour Angle.

Here we have given the three sides of the triangle P Z S, to find the angle at P.   We use in this case formulæ (3) and (4), letting Z P S, (t), be the angle A.   We can then change these formulæ so as to make them more convenient in practice.

Let $s_1 = \frac{1}{2}(\delta + \phi + z)$, then

$s_1 - z = \frac{1}{2}(\delta + \phi + z) - z = \frac{1}{2}(\delta + \phi - z)$

Formula (4) gives
$$s = \tfrac{1}{2}(90° - \delta + 90° - \phi + z) =$$
$$90° - \tfrac{1}{2}(\delta + \phi - z)$$
But $s_1 - z = \tfrac{1}{2}(\delta + \phi - z)$, therefore,
$$s = 90° - (s_1 - z)$$
In the same way we may show that
$$s - a = 90° - s_1$$
$$s - b = s_1 - \phi$$
$$s - c = s_1 - \delta$$
Substitute these in (3) and we obtain
$$(20) \qquad \text{tg } \tfrac{1}{2}t = \sqrt{\frac{\sin(s_1 - \phi)\ \sin(s_1 - \delta)}{\cos s_1 \cos(s_1 - z)}}$$
This together with
$$(21) \qquad s_1 = \tfrac{1}{2}(\delta + \phi + z)$$
should be used in the above case rather than (3) and (4).

Remarks on Methods of Computation

The student's attention is called to the form in which the problems on the opposite page are worked. It frequently happens in practice that he will be called upon to solve a number of problems similar in every respect except the numerical values of the quanitites employed. Thus he may observe ten altitudes of the sun and wish to reduce each separately. By following such a form much time is saved and errors are guarded against. The following precepts are given for his guidance.

1. Always compute with pen and ink.

2. Prepare a form as above and fill out column I before any numerical work is done. The proper filling out of column I will make it certain that he understands the theory of his problem and when he comes to the numerical work he can devote his entire attention to it.

3. Compute horizontally and not vertically, i. e., solve

GIVEN:— $h_1 = 29° 17' 28''$ $A_2 = 302° 45' 38''$
$h_2 = 32° 18' 21''$ $A_2 = 297° 40' 32''$
$\phi = 39° 25' 22''$

REQUIRED:— δ_1, δ_2, t_1, and t_2.

I	II	III
h	$29° 17' 28''$	$32° 18' 21''$
A	$302° 45' 38''$	$297° 40' 32''$
cot h	0.25106	0.19906
cos A	9.73330	9.66695
Sum = tg Z	9.98436	9.86601
Z	$43° 58' 07''$	$36° 17' 55''$
* $\phi - Z$	$- 4° 32' 45''$	$+ 3° 07' 27''$
sin Z	9.84153	9.77232
$\cos(\phi - Z)$	9.99863	9.99936
§ Difference	9.84290	9.77296
tg A	0.19146_n	0.28028_n
Sum = tg t	0.03436_n	0.05324_n
‡ t in ° ' ''	$312° 44' 09''$	$311° 29' 48''$
t in h. m. s.	$20^h 50^m 56^s.6$	$20^h 45^m 59^s.2$
$tg(\phi - Z)$	8.90041_n	8.73704
cos t	9.83163	9.82123
Sum = tg δ	8.73204_n	8.55827
† δ	$- 3° 05' 18''$	$+ 2° 04' 16''$
cos h	9.94059	9.92696
cos A	9.73330	9.66695
Sum_1	9.67389	9.59391
cos δ	9.99937	9.99972
cos t	9.83163	9.82123
Sum_2	9.83100	9.82095
$Sum_1 - Sum_2$, check.	9.84289	9.77296

NOTE:— For the notes on this example, see pages 16 and 17.

all the problems at the same time and do not finish the first, then the second, and so on. In this way much time will be saved in turning over the pages of tables.

4. Never write a number unless it has a name, i. e., unless some statement precedes the number to show what it represents.

5. Where quantities have the same values in each of a series of problems they should be written on a separate slip of paper and carried along from column to column. Thus in the above example, to compute $\phi - Z$, where ϕ is the same and Z different in each problem, write the value of ϕ on a slip of paper and run it along above the Z line forming the differences as you go.

6. Learn to subtract from left to right.

7. In picking out the functions of angles greater than 90° always go forward and never backward; thus if the $\sin(297° \, 13' \, 45'')$ is required, pick out the $-\cos(297° \, 13' \, 45'' - 270'') = -\cos(27° \, 13' \, 45'')$. This can be done mentally while if we went backward we would have the $\sin(297° \, 13' \, 45'') = -\sin(360° - 297° \, 13' \, 45'')$, which subtraction could not be performed mentally. The same method should be employed in finding the angle corresponding to any given function. Thus to find the angle whose log tg $= 9.27514_n$ and whose cos is $+$, we pick out the angle, less than 90°, whose log cot $= 9.27514$, and add 270° since, the tangent being negative and the cosine positive, the angle must be in the fourth quadrant. The following rule will help. Let A be an angle greater than 90° and a an angle less than 90°, such that we may write the following equation, $A = n \times 90° + a$. Then when n is odd take the co-function and when n is even take the same function.

*See precept 5.

§ This is the check number, and should agree within one or two units of the last decimal place with the last line of the problem.

‡ h and δ are always numerically less than 90°, hence formula (9) shows us that sin A and sin t must have the same sign. Tg t is — and sin A is — also, therefore sin t is — and hence t must be in the fourth quadrant.

† δ always has the same sign as tg δ.

The above notes refer to the example on page 15.

PROBLEMS

(1) In the spherical triangle whose vertices are the pole, the star and the pole of the ecliptic, what are the sides and angles?

(2) Show that if t and δ are given we may compute h and A by the following formulæ:—

(22) $\cot Z = \cot \delta \cos t$ (23) $\operatorname{tg} h = \cot(\phi - Z)\cos A$

(24) $\operatorname{tg} A = \dfrac{\operatorname{tg} t \cos Z}{\sin(\phi - Z)}$ (25) $\dfrac{\cos \delta \cos t}{\cos h \cos A} = \dfrac{\cos Z}{\sin(\phi - Z)}$

(3) Show that if a and δ and the angle (e) between the ecliptic and the equator are given, we may compute L and B by the following formulæ:—

(26) $\cot Z = \cot \delta \sin a$ (27) $\operatorname{tg} B = \operatorname{tg}(Z - e) \sin L$

(28) $\operatorname{tg} L = \dfrac{\cos(Z - e)}{\cot a \cos Z}$ (29) $\dfrac{\cos B \sin L}{\cos \delta \sin a} = \dfrac{\cos(Z - e)}{\cos Z}$

(4) Given $t = 22^h 29^m 17^s.5$, $\delta = + 28° 14' 47''$ and $\phi = 39° 43' 36''$. Find h and A.

(5) Given $a = 15^h 38^m 19^s.8$, $\delta = - 36° 42' 53''$ and $e = 23° 47' 22''$. Find L and B.

(6) Given $z = 38° 34' 44''$, $\delta = + 32° 29' 09''$ and $\phi = 39° 56' 55''$. Find t.

(7) Given $h = 21° 09' 19''$, $A = 309° 19' 18''$ and $\phi = 41° 01' 09''$. Find t and δ.

CHAPTER III.

When a ray of light passes from a rarer medium, such as air, to a denser medium, such as glass, it is bent towards the normal to the surface of separation of the two media. This bending is called refraction and is subject to the following laws:— 1st. The two rays and the normal to the surface lie in the same plane. 2nd. The sine of the angle the incident ray makes with the normal to the surface bears a constant ratio to the sine of the angle the refracted ray makes with this same normal. This second law may be expressed by the equation, $\sin i = n \sin r$. The constant n is called the index of refraction.

The earth's atmosphere is optically denser than the ether, and extends upward, in ever decreasing density, to a distance of about 45 miles. When, therefore, a ray of light from a star reaches this atmosphere it must be bent towards the earth's surface; and since the density increases inward, this bending will be gradual, and the path of the ray inside the atmosphere will be curved. The direction in which we see the star will be that of the tangent to this curved path at the eye of the observer. The refraction is the angle between this tangent and the original direction of the ray outside of the atmosphere.

Since the normal to the surface of the atmosphere and the path of the ray of light must lie in one plane, refrac-

tion must all take place in altitude and none in azimuth. The theoretical determination of this angle from the known constants of the air, is quite complex and beyond the limits of these notes. It has been found however, that for a standard condition of the air the refraction can be represented very closely by the following law: $r_0 = 57''.7$ tg z, r_0 being the mean refraction. Up to $z = 70°$, the above law gives r_0 to within $1''$; for more accurate values r_0 must be taken from Table I given in the appendix.

The constant $57''.7$ must depend, among other things, upon the index of refraction of air; and, since this in turn depends upon the density of the atmosphere, whatever affects this latter must change the constant $57''.7$. Now the density of the atmosphere varies with the temperature and the barometric pressure. Every mercurial barometer has attached to it a small thermometer which gives the temperature of the mercury column, and the readings of the barometer must be corrected for this temperature.

From the above considerations it is evident that refraction must depend upon four quantities, namely; the star's zenith distance (z), the temperature of the air (T), the reading of the barometer (B) and the reading of the attached thermometer (t). Theory, confirmed by observation, shows that the refraction for any state of the atmosphere, e. i., for any values of the quantities T, B and t, will be given by $r = r_0 \times \psi_1(T) \times \psi_2(B) \times \psi_3(t)$, where r_0 is the mean refraction, e. i., the refraction for a standard set of values of T, B and t, and ψ denotes a function of the quantities within the parentheses. The logarithms of these functions are given in Tables II, III and IV in the appendix.

The sun, moon and planets appear in a different di-

rection as seen from the surface of the earth, than they would appear if seen from its center. It is necessary, therefore, to correct an observed altitude of these bodies in order to reduce it to the center of the earth. With all bodies except the moon we may consider the earth as a perfect sphere.
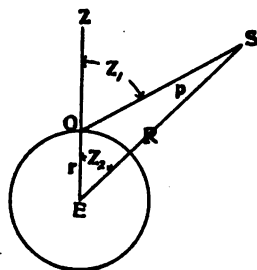
FIG. II

Let E, FIG. II, be the center of the earth, O the observer, Z the zenith, and S the sun.

Then the angle Z_1 will be the observed zenith distance and the angle Z_2 will be the zenith distance of the sun as seen from the center of the earth. The difference $Z_1 - Z_2$ is called the parallax of the body S. From the triangle SOE we have the following relations: $p = Z_1 - Z_2$ and $R \sin p = r \sin Z_1$. Since p is always small we may replace sin p with p, whence we obtain $p = (r \div R) \sin Z_1$. When $Z_1 = 90°$, and the observer is at the equator, the parallax becomes equal to the ratio $r \div R$ and is called the equatorial horizontal parallax. We will denote this quantity by P, whence we have

(30) $p = P \sin Z_1 = P \cos h$

In the case of the sun, the value of P changes from day to day and is given in the American Ephemeris for every ten days in the year; for most purposes, however, its mean value of $8''.85$ may be used.

It should be noted that refraction and parallax always apply with opposite sign.

To illustrate the above theory, we have the following example:— The altitude of the sun was observed with a

theodolite and found to be 22° 19′ 15″.5, the barometer read 29″.22, the attached thermometer read 80° F., and the external temperature was 56° F. Find the true altitude of the sun.

FIRST:— REFRACTION CORRECTION

From table I, with the argument 22° 19′
we find $r_0 = 2′\ 21″.0 = 141″.0$, whence $\log r_0 = 2.1492$
From table II, with the argument 56° F.
we find $\log \psi_1(56°\ F.) = \ldots \ldots \ldots \ldots$ 9.9949
From table III, with the argument 29″.22
we find $\log \psi_2(29″.22) = \ldots \ldots \ldots \ldots$ 9.9886
From table IV, with the argument 80° F.
we find $\log \psi_3(80°\ F.) = \ldots \ldots \ldots \ldots$ 9.9989
Whence $\log r$ = the sum = $\ldots \ldots \ldots$ 2.1316
Therefore, $r = 135″.4 = 2′\ 15″.4$, and the altitude corrected for refraction = 22° 17′ 00″.1

SECOND:— PARALLAX CORRECTION

$\log 8″.85 = \ldots \ldots \ldots \ldots \ldots \ldots$ 0.9469
$\cos(22°\ 17′) = \ldots \ldots \ldots \ldots \ldots$ 9.9663
$\log p$ = sum = $\ldots \ldots \ldots \ldots \ldots$ 0.9132
Whence $p = 8″.2$, and the true $h = 22°\ 19′\ 15″.5 - 2′\ 15″.4 + 8″.2 = 22°\ 17′\ 08″.3$

CHAPTER IV.

TIME

Time is one of the three fundamental physical quantities which, though we all know what it is, yet can not be defined. We may, however, define the unit in which it is measured. This unit is the time it takes the earth to revolve on its axis and is called the day. In order to be able to tell when the earth has made one complete revolution, we must have some point disconnected with the earth, whose transit across some fixed line on the earth's surface may be noted. Astronomers adopt the meridian as the fixed line. Two points naturally suggest themselves as fixed points in space, whose transits across the meridian are to be noted. These are the sun and some bright star.

If we note on some good clock, the interval between two consecutive transits of the sun across the meridian, and also the interval between two consecutive transits of a star, we would find that it takes the sun about four minutes longer, after crossing the meridian to come back to it again than it takes the star. We would find also that this interval is the same for all stars. In other words the unit of time is different according as we use the sun day or the star day. Having two units we must give them different names; the first is called the apparent solar day and the second, the sidereal day.

Again, if we used our clock to measure the length of the apparent solar day at different times during the year, we would find that this is not a constant quantity, but is sometimes longer and sometimes shorter, depending upon the time of the year at which we performed our experiment. Trying this same experiment with a star we would find that the sidereal day is absolutely constant.

Since units are always supposed to be invariable, we would naturally adopt the sidereal day as the standard; but we have seen that this gains about four minutes daily on the apparent solar day, and, though this is a small quantity, yet in six months it would amount to twelve hours. So if we went by the stars we would literally turn night into day every six months.

In order to avoid the difficulty of using the real sun to measure our time, astronomers have invented an imaginary or mean sun, whose consecutive transits give us the mean solar day. The length of this mean day is taken as the average or mean of all the apparent solar days.

There are then in astronomy three kinds of days, and as a consequence three kinds of time. We may define the apparent solar time as the interval which has elapsed since the true sun was last on the meridian. This is the time kept by the sun-dial. The mean solar time is the time which has elapsed since the mean sun was last on the · meridian. This is the time in common use that is kept by our clocks and watches. The sidereal time is the interval which has elapsed since the vernal equinox was last on the meridian. This is the time kept by the stars and the so-called sidereal clocks and chronometers found in observatories. Astronomers use the vernal equinox, rather than a star, as it is one of the fundamental points of the co-ordinate systems; and, moreover, each star has

a real, though very small, motion of its own.

Since the earth turns uniformly on its axis, the angle at the pole, between the meridian and the hour circle through any star, is proportional to the time which has elapsed since the star was on the meridian. This angle is the star's hour angle; hence a star's hour angle measures the time since the star was on the ·meridian. If, therefore, we express hour angles in hours, minutes, and seconds, we may say that the apparent solar time is the hour angle of the true sun, the mean solar time is the hour angle of the mean sun, and the sidereal time is the hour angle of the vernal equinox.

This method of measuring an angle with a clock rather than a graduated circle seems at first somewhat difficult to understand. The reverse problem of measuring time with a graduated circle is in use on every clock. The dial is the graduated circle and the hands are the verniers or pointers. So all we have to do to complete the analogy is to think of the north pole of the earth as the axis to which the hands of our earth-clock are fastened, the meridian as the hour-hand, and the stars as the graduated circle or dial-plate. The time can then be read from the meridian as it sweeps over this celestial dial-plate, where in place of the usual black figures we find glittering points of light — the stars.

We will now examine more closely the relations which
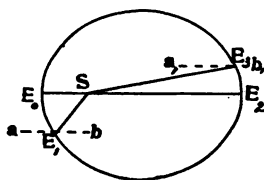
FIG III.

exist between these three kinds of time. Let E_0 E_1 E_2 E_3, Fig. III, be the earth's orbit, with the sun at one of the foci, S.

Suppose the sun makes one complete revolution on its axis in passing from E_0 to E_1 and also

in passing from E_2 to E_3. It will, therefore, take the earth just as long to pass from E_0 to E_1 as it does to pass from E_2 to E_3. From Newton's law of gravitation it can be shown that the areas $E_0 S E_1$ and $E_2 S E_3$ must be equal. Suppose the sun is directly overhead when the earth is at E_0; before the sun comes overhead again the earth will not only have to make one complete revolution, but in addition turn through the angle $b E_1 S$, or rather this angle and a little more since the earth keeps on moving in its orbit. At E_2 the earth will turn through $a_1 E_3 S +$ a complete revolution and a little more to compensate for the earth's continued motion. Now, obviously from the above law of areas, $a_1 E_3 S$ must be less than $b E_1 S$ and the apparent solar day will be longer at E_0 than it will be at E_2. In other words the sun does not move uniformly along the ecliptic.

However, even if the sun did move uniformly along the ecliptic, since the ecliptic and the equator are inclined at an angle of $23\frac{1}{2}°$, uniform motion along the ecliptic would not correspond to uniform motion along the equator and as a consequence to equal increments of the sun's hour angle. These two causes produce the variation in the length of the apparent solar day. Astronomers, therefore, in order to remedy this inequality, have invented an imaginary sun called the mean sun, which moves uniformly along the equator and completes the entire circuit in the course of a year.

The difference in the length of the apparent solar days is small, yet it may accumulate so that mean solar and apparent solar time may differ by as much as 16 minutes. This difference is called the equation of time. A stricter definition is as follows:— The equation of time is that quantity which must be added algebraically to the appar-

ent time in order to obtain the mean time. Its value is given in the American Ephemeris for Washington apparent noon of each day of the year.

The earth revolves on its axis in the same direction as it revolves about the sun. Now suppose the earth made only one revolution on its axis in the course of a year; the rotation on its axis would be exactly compensated by its rotation around the sun and the sun would appear stationary. This is because the earth goes around the sun as it goes around all points inside of its orbit. It does not, however, go around points outside of its orbit, hence its orbital motion can not undo for such points the effect of its rotation on its axis. Not only does the vernal equinox lie outside of the earth's orbit but it is also at an infinite distance; we may, therefore, so far as the vernal equinox is concerned, neglect entirely the effect of the earth's orbital motion, and, under the above supposition, the vernal equinox would appear to make one complete revolution about the earth. From this it follows, that the vernal equinox will appear to make just as many revolutions yearly about the earth as, in reality, the earth makes revolutions on its axis, but the sun will always appear to make one revolution less. The earth makes 366.24222 revolutions on its axis in the course of a year, and therefore the vernal equinox will transit the meridian 366.24222 times but the sun only 365.24222 times.

From the preceding paragraph we see that there will be 366.24222 sidereal and 365.24222 mean solar days in a year. In order, therefore, to convert any interval measured in mean time h. m. s. into the corresponding interval measured in sidereal h. m. s., we must multiply the mean time h. m. s. by $\dfrac{366.24222}{365.24222}$; and to convert an interval

measured in sidereal h. m. s. into the same interval meas-
ured in mean time h. m. s., we must multiply the sidereal
h. m. s. by $\dfrac{365.24222}{366.24222}$. This process may be made more
convenient in practice as follows:— The two fractions
$\dfrac{366.24222}{365.24222}$ and $\dfrac{365.24222}{366.24222}$ are equal respectively to the
numbers 1.0027379 and 0.9972696, which in turn may be
written (1 + 0.0027379) and (1 — 0.0027304). Hence,
if S be the sidereal time interval and M be the mean time
interval, we can write S = (1 + 0.0027379) M = M +
0.0027379 M and in the same way M = S — 0.0027304 S.
The values of 0.0027379 M and 0.0027304 S are given in
Tables II and III found in the back part of the American
Ephemeris.

The above equations are equivalent to the following
rules. To convert a sidereal interval into the correspond-
ing mean time interval: enter Table II with the sidereal
interval as argument, subtract the quantity there found
from the sidereal interval, and the result will be the mean
time interval. To convert a mean time interval into the
corresponding sidereal interval: enter Table III with the
mean time interval as argument, add the quantity there
found to the mean time interval, and the result will be
the sidereal interval.

We are now ready to derive the rules for the conver-
sion of mean solar time at a given date into sidereal time
and the converse. These problems must be distinguished
from the preceding as here the date is specified. Two
examples will illustrate the difference. The above rules
enable us to answer such questions as this:— How many
sidereal h. m. s. are equivalent to six mean time hours?
We are now to derive the rules to answer such a question

as this:— What is the sidereal time corresponding to July 10th, 1897, 5^h 19^m 39^s mean time?

In Fig. I, page 6, the arc VBG is the same as the hour angle of the vernal equinox, or in other words it is the sidereal time. Now if S be the mean sun the arc BC, equal to the angle BPC, is the hour angle of the mean sun, e. i., the mean time. If we imagine the sun S to move backward, carrying the hour circle PSB and the vernal equinox with it, until PSB shall coincide with the meridian, we will then have the sun on the meridian, e. i., it will be mean noon. VB then becomes the hour angle of the vernal equinox at the instant of mean noon, or in other words the sidereal time of mean noon. It would seem, therefore, that since $VC = VB + BC$, all we would have to do would be to add the sidereal time of mean noon to the mean time and we would have the sidereal time. We must not forget, however, that BC is measured in mean time h. m. s. while VC is measured in sidereal h. m. s., and we can no more add the two together than we can add inches and millimeters. Before this addition can be made the mean time interval must be converted into a sidereal interval by the methods given above.

The sidereal time of mean noon is given in the American Ephemeris for the meridian of Washington for each day of the year. In the majority of cases the observer will be located elsewhere and will need the sidereal time of mean noon for his own locality or longitude. To correct this for longitude, we note that the sidereal time of mean noon increases regularly by about 3^m $56^s.55$ per day. Therefore to correct for longitude we must add to the sidereal time of mean noon given in the Ephemeris the quantity $(3^m\ 56^s.55) \times L = (236^s.55) \times L$, where L denotes the longitude expressed in decimals of a day. Table

III, as we have seen, is a table of proportional parts and gives 3^m 56^s.556 for the argument 24 hours. We may, therefore, correct the sidereal time of mean noon for longitude by entering Table III with the longitude as argument and adding the quantity there found to the sidereal time of mean noon as taken from the Ephemeris.

It is customary with astronomers to begin the astronomical day at noon of the corresponding civil date and to count continuously up to 24 hours. Thus July 21th, 10 A. M., civil reckoning, would be July 20th, 22^h. It is also customary, when writing any instant in sidereal time, to prefix the date in civil reckoning. We do this because otherwise, since there are 366 sidereal days in a year, at some time during the year we would be compelled to introduce an additional day. For example, February 1st, 1894, 3^h 2^m 0^s Washington mean time corresponds to February 1st, 1894, 23^h 50^m 3^s.17 sidereal time; and February 1st, 1894, 4^h 17^m 22^s Washington mean time corresponds to February 1st, 1894, 1^h 5^m 37^s.55 sidereal time. Thus the zero point of the sidereal day on February 1st, 1894, occurs between 3 and 4 o'clock P. M. and we would naturally, perhaps, write the second date as February 2nd., which would be wrong.

We may now collect the results of the preceding paragraphs into the following rules:—

1. Mean Solar into Sidereal Time.

1. Enter Table III with the longitude as argument, add the quantity there found to the sidereal time of mean noon for the given date as taken from the Ephemeris. The result will be the local sidereal time of mean noon.

2. Enter Table III with the mean time as argument, add the quantity there found to the mean time, and the

result will be the sidereal interval since mean noon.

3. Add the local sidereal time of mean noon as given by 1 to the sidereal interval since mean noon as given by 2, and the result will be the sidereal time.

2. Sidereal into Mean Solar Time.

1. Find the local sidereal time of mean noon as in 1 of the preceding rule.

2. Subtract this from the given sidereal time. This gives the sidereal interval since mean noon.

3. With the sidereal interval since mean noon as argument, enter Table II and subtract the quantity there found from the sidereal interval since mean noon, and the result will be the mean time.

3. Apparent Solar into Mean Solar Time.

1. To the local apparent time add the longitude west of Washington and express the result in decimals of a day. This gives the Washington time expressed in decimals of a day. Three places will generally be sufficient.

2. Pick out the value of the equation of time (E) for Washington apparent noon for the given date and the change in its value in 24 hours.

3. Multiply the daily change in E by the Washington apparent time in decimals of a day. Add the result to the value of E as given by 2. The result is the value of E for the given instant.

4. Add the value of E as found in 3 to the local apparent time and the result will be the local mean time.

4. Mean Solar into Apparent Solar Time.

1. Add the local mean time to the longitude west of Washington and subtract from this the value of E given

in the Ephemeris for Washington apparent noon. The result will be the Washington apparent time very nearly.

2. Convert the Washington apparent time as given in 1 into decimals of a day and find E as in 2 and 3 of rule three.

3. Subtract the value of Ė as found by 2 from the local mean time and the result will be the local apparent time.

Note:— In the preceding rules, longitudes east of Washington are minus and those west are plus.

We have seen above that the sidereal time is the same as the distance measured along the equator from the vernal equinox to the meridian. A star's right ascension is also defined as the distance from the vernal equinox to the foot of the hour circle through the star. Obviously, therefore, a star's right ascension may be defined as the sidereal time when it is on the meridian. Again in FIG. I, page 6, VB is the star's right ascension (a), BC is the star's hour angle (t) and VC is the sidereal time (T); whence we have the equations

$$T = a + t \text{ and } t = T - a$$

The following examples will illustrate the preceding rules:— At a place whose longitude is $6^h 18^m 19^s.13$ west, the sun's hour angle was found to be $3^h 19^m 17^s.14$ on September 15th, 1894; find the hour angle of the star a Lyræ. In order to solve this problem, we note that the sun's hour angle is the same as the apparent time and, if we have the corresponding sidereal time, all we need to do is to subtract the right ascension of a Lyræ from it in order to get the hour angle of a Lyræ. In other words, our problem consists in converting apparent solar time into mean solar time and this latter into sidereal time. The numerical solution is as follows:—

Example No. 1

h. m. s.

Local apparent time 3 19 17.14
Longitude west of Washington 6 18 19.13
Sum = Washington apparent time 9 37 36.27
9^h 37^m $36^s.27$ = 0.401 days.
From the Ephemeris for 1894, page 382, we
find the change in E in 24 hours . . . . — 21.29
ΔE = — $21^s.29$ × 0.401 = ... — 8.54
The Ephemeris, page 382, gives E for Sep-
tember 15th, Washington apparent noon . . . — 4 57.19
E at instant required = sum . . . . — 5 05.73
Local apparent time 3 19 17.14
Local mean time = sum 3 14 11.41
From the Ephemeris, page 382, we find the
sidereal time of Washington mean noon on
September 15th 11 38 34.93
From Table III, page 529, with the longi-
tude (6^h 18^m $19^s.13$) as argument, we find
the correction for longitude 1 02.15
Sum = local sidereal time of mean noon . . 11 39 37.08
Local mean time 3 14 11.41
From Table III, with the local mean time as
argument, we find 31.90
Local sidereal time = sum 14 54 20.39
From the Ephemeris, page 351, we find the
right ascension of a Lyræ 18 33 22.78
Difference = the hour angle of a Lyræ . . — 3 39 02.39
The minus sign denotes that the star is east of the me-
ridian.

Example No. 2

At a place whose longitude is 5^h 49^m $13^s.45$ east of
Washington, the hour angle of a Lyræ was found to be

4^h 32^m $27^s.68$ west on January 15th, 1901; find the sun's
hour angle at the same instant. It is to be noted that
this is the converse of Example No. 1. The numerical
solution is as follows:—

                                              h. m.   s.
From page 381 of the Ephemeris for 1901,
we find the right ascension of α Lyræ . . . 18 33 34.29
The hour angle of α Lyræ is 4 32 27.68
Sum = local sidereal time 23 06 01.97
Sidereal time of local mean noon as in the
preceding example 19 36 51.12
Difference = the sidereal interval since
mean noon 3 29 10.85
From Table II, page 585 of the Ephemeris
for 1901, with 3^h 29^m 11^s as argument . . . , . . 34.27
Difference = the local mean time 3 28 36.58
Longitude . . . . — 5 49 13.45
Washington mean time January 14th . . . 21 39 23.13
From the Ephemeris for 1901, page 400, we
find the change in E in 24^h = + $21^s.66$
and also the value of E itself for Washing-
ton apparent noon January 15th . . . . . . + 9 33.41
Difference = the Washington apparent time
approximately 21 29 49.72
January 14th 21^h 29^m 50^s = January 15th
minus 0.104 days. ΔE = + $21^s.66$ × — 0.104 = — 2.25
Whence E for the instant required is . . . . + 9 31.16
Local mean time 3 28 36.58
Difference = the sun's hour angle 3 19 05.42

PROBLEMS

(1) On July 4th, 1901, a sidereal chronometer whose
ΔT was + $33^s.25$ on local time at a place whose longitude

is 9^h 19^m $17^s.53$ west, was found to read 3^h 19^m $12^s.00$ at the same instant that a mean time chronometer read 20^h 33^m $50^s.00$. What is the error of the mean time chronometer on local mean time?

(2) Given the following apparent times, find the corresponding mean times.

DATE	LONGITUDE	APPARENT TIME
1901	h. m. s.	h. m. s.
December 21st . . .	12 14 17.98 west . .	11 09 19.88
July 9th	3 15 34.18 east . .	4 09 33.01
May 8th	9 48 16.40 east . . .	6 34 56.78
January 30th	8 54 32.10 west . .	21 48 13.89
November 14th . . .	5 21 16.12 west . .	2 03 14.09

(3) At what times during the year will the sidereal time of mean noon have the following values:— 3^h 22^m? 16^h 19^m? 22^h 15^m? Solve without the Ephemeris and assume that the sidereal time of mean noon is zero on March 21st at noon.

(4) If the earth's axis were perpendicular to the ecliptic, how many times in the course of a year would the Equation of Time become zero and what would be the position of the earth in its orbit at these times?

(5) Using the data of Example 1, find the local mean time when α Lyræ is on the meridian.

(6) Assume the times given in Problem 2 are local mean times, find the corresponding apparent times.

(7) Find the sidereal times corresponding to the apparent times given in Problem 2.

(8) A sun-dial at St. Louis was 2^m $22^s.15$ fast of a watch on July 1st, 1901. The watch read 3^h 19^m $22^s.00$. Find the error of the watch on local mean time, assuming that St. Louis is on the 90th meridian and that Washington is 5^h 08^m $15^s.78$ west of Greenwich.

(9) A mean time and a sidereal clock, both beating seconds, were in unison at 7^h 15^m 17^s A. M. by the mean time clock whose ΔT on local mean time was — $29^s.15$. The sidereal clock read 20^h 15^m 19^s. If the longitude of the place is 5^h 19^m 18^s east and the date is April 4th, 1901, what is the error of the sidereal clock on sidereal time?

(10) Assuming that the right ascension of Polaris is 1^h 22^m $58^s.51$ during the year 1901, what will be the mean time when it is be on the meridian both above and below the pole on August 22nd, October 24th, May 2nd, November 27th and February 14th?

(11) Given the following sidereal times, find the corresponding mean times.

DATE 1901	LONGITUDE h. m. s.	SIDEREAL TIME h. m. s.
May 22nd	2 14 17.15 west	5 19 22.55
July 14th	11 09 14.39 east	23 14 14.22
February 23rd	3 21 46.72 west	4 19 17.18
March 14th	9 16 27.55 west	18 27 14.76
September 23rd	10 22 27.18 east	9 19 38.76

(12) At St. Louis on April 14, 1901, a sidereal chronometer read 4^h 54^m $29^s.19$ when the sun's hour angle was $50°$ $19'$ $14''.3$ west of the meridian. The chronometer was then carried to a new place and on April 17th, civil reckoning, read 19^h 15^m $35^s.03$ when the sun's hour angle was $62°$ $15'$ $22''.9$ east of the meridian. Assuming that the chronometer gains $10^s.48$ a day on chronometer time, find the longitude of the new place. The longitude of St. Louis is given in Problem 8.

Problem 9 illustrates a method by which a mean time and a sidereal chronometer may be compared within $0^s.03$ or $0^s.04$. The observer listens to them until they beat in unison and records their simultaneous readings.

CHAPTER V.

REDUCTION OF STAR PLACES

We have seen in Chapter I that two of the co-ordinate systems depend upon the planes of the equator and of the ecliptic. The zero point in both systems is the vernal equinox. If these planes remain constantly parallel to themselves as time goes on, the corresponding coordinates will remain constant. Such, however, is not the case, as the normal to the equator, e. i., the earth's axis, makes a revolution about the normal to the ecliptic once in about twenty-six thousand years. This causes the vernal eqinox to slip backward along the ecliptic at the rate of about $50''$ per annum. This backward motion is called precession.

The causes of precession are not hard to understand, though their mathematical investigation is very difficult and far beyond the limits of this book. A most beautiful counterpart of this majestic march of the vernal equinox through the stars is found in an ordinary peg top, which gives us an illustration and qualitative explanation of this phenomenon. Two facts must be observed; first, that the top will not stand up unless it is spinning and second, that when spinning, unless the axis about which it turns, the so-called axis of spin, is exactly vertical, the axis of spin slowly rotates about the plumb line. Such motion of a spinning body is called precession and its axis of

spin is said to precess. We may note then that the top's precession depends not only upon its spinning but also upon the action of a force or couple which, when the spinning stops, causes the top to fall on its side.

In order to see what these forces are let us examine FIG. IV. Here we have W the weight of the top acting downwards, R the resistance of the plane on which the top is spinning acting upwards, AD the plumb line and AE the axis of spin. These forces, since they do not pass through the same point, form an unbalanced couple acting so as to rotate the axis of spin into the plane on which the top is spinning. An unbalanced couple must produce rotation. Since the top is spinning, the inertia of the rapidly moving body transforms the direct action of the couple so that in place of rotating the axis of spin into the plane on which the top is spinning, it causes the axis of spin to precess about the plumb line.
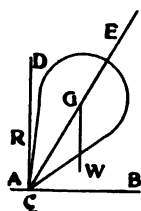
FIG IV.

Now if we look at our spinning toy carefully, we will see that the direction of its precession is the same as that of its spin. If one is clock-wise the other is also. With a little practice we can wind one top right-handed and the other left-handed, then throwing one with the right hand and the other with the left hand, we will see our two tops one spinning and precessing clock-wise and the other counter clock-wise. This equality of direction is due to the fact that the couple which causes precession acts so as to turn the axis of spin into the plane on which the top is spinning. If we could make a top in which this couple acted so as to rotate the axis of spin away from the plane on which the top is spinning, we would find this equality of direction changed into an

inequality; in other words, under that condition a top spinning clock-wise would precess counter clock-wise and one spinning counter clock-wise would precess clock-wise.

Now can we find an analogous state of affairs in the case of the earth? We know that it is spinning and that its axis is inclined by $66\frac{1}{2}$ degrees to the ecliptic, the plane on which it is spinning. Can we find a couple which tends to change this inclination? If so, our top tells us that the earth's axis must precess.

The earth is an ellipsoid of revolution revolving about the minor axis. In FIG. V, let O be the center of the
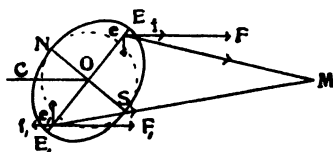
FIG. V.

earth, EE_1 the equator, NS the earth's axis and M the center of the moon. Let the dotted circle represent the inscribed sphere. Now since this sphere is symmetrical about O, we may eliminate it from this elementary treatment of the problem and consider only the moon's attraction upon the equatorial bulges at e and e_1.

Newton's law of gravitation tells us that every particle of this bulge is attracted towards the center of the moon with a force that varies inversely as the square of the distance. Thus at e we have a force in the direction eM and at e_1 we have a lesser force in the direction e_1M. The earth is kept from falling into the moon by its centrifugal force at O directed away from M. Let us resolve these forces into their components parallel and perpendicular to the line OM. The components, (F and F_1), parallel to OM are exactly balanced by the centrifugal force F_2. We may, therefore, subtract $\frac{1}{2}F_2$ from each of these and we will have left at e, $F - \frac{1}{2}F_2 = f$ parallel to OM and in the direction OM, while at e_1 we will have

left $\frac{1}{2}F_2 - F_1 = f_1$, parallel to OM and in the direction MO. In addition to these two parallel components there will be two components perpendicular to the line OM and both directed towards it.

These four forces give us our couple and a glance at the figure shows that they tend to rotate the axis of spin into the normal to the plane of the earth's orbit. But our earth top is spinning, and, if we imagine ourselves at the north pole looking down upon it, is spinning counter clock-wise; hence the earth's axis must precess clock-wise and, carrying the equator with it, must cause the vernal equinox to slip backwards through the stars.

The moon though the nearest is not the only body that can thus get a grip on the earth's equatorial bulge. The sun though vastly more remote has about twenty-seven million times as much mass and, therefore, causes a similar action. The sum of these two is called the luni-solar precession and causes the plane of the equator to slip backwards on the plane of the ecliptic, thus increasing a star's longitude without affecting its latitude and hence changing both right ascensions and declinations.

The planets also come in for their share, which is, however, somewhat different. The planetary precession, as it is called, causes the plane of the ecliptic to slip on the equator. This is much smaller than the luni-solar precession and changes right ascensions without affecting declinations. The sum of these two is called the general precession and is what must be taken into account in reducing a star's co-ordinates from one date to another. Before giving the methods by which this is done in practice, we will examine some other causes which affect a star's co-ordinates.

We have seen that precession depends chiefly upon

the attraction of the sun and moon, but the sun and moon
are constantly changing their relative position and as a
consequence this attraction must be constantly changing
not only in magnitude but also in direction. It is as if
we were to spin an iron top upon an electromagnet; as
we would vary the current strength, we would vary the
force which causes the top's precession, and we would
find the axis of spin nodding back and forth about its
mean position in complete harmony with the varying
strength of the magnet. Similarly the earth's axis nods
back and forth about its mean position, and this nodding
motion is called nutation. The sun and moon come to
practically the same position once every nineteen years,
hence nutation must be periodic and its period will be
nineteen years. The mathematical investigation of this
is exceedingly difficult, but the application of the final
formulæ is made very easy by the aid of certain tables
published in the American Ephemeris, the application of
which will be given later on.

A third correction to a star's co-ordinates, called ab-
erration, must now be explained. Every reader of this
book knows that, if the rain is falling straight down and
he starts to run, he will have to tilt his umbrella forward,
in order to keep off the rain. In other words, the point
from which the rain seems to fall is shifted in the direc-
tion of his running. Every telescope on the earth is mov-
ing with a velocity of nineteen miles a second, and the
waves of light move forward, like the rain drops, in
straight lines; and in order to catch them the telescope
must be tilted in the direction of the earth's motion.

In order to examine this more in detail, let us refer
to Fig. VI, page 41, where ABCD is a telescope in the
position it has at the instant of time T at which light from

the star reaches the objective, and A′B′C′D′ the position it has at the instant of time T′ when light, converged by the objective, reaches the focus f. The direction in which any object is seen in a telescope is given by the line which joins its image on the focal plane to the center of the objective. Now since the path of the light rays is unaffected by the motion of the objective, the image of the object will be formed at the same point f, whether the telescope is in motion or at rest. If the telescope had been still the star would have been seen in the direction fO, O being the center of the objective at the time T; but, during the interval T′ — T, the center of the objective has moved to a new position O′, hence the star will appear in the direction fO′. The angle OfO′ = a is the amount the star appears shifted from its true position and is called the aberration.
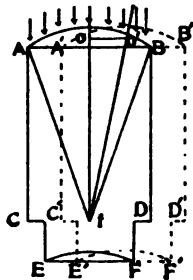
FIG. VI.

In order to calculate the angle a, let V equal the velocity of light and v the velocity of the telescope. While light passes from O to f, the center of the objective moves from O to O′, the time required being the same and equal to T′ — T; evidently, therefore,

$$OO' = (T' - T)v \text{ and } Of = (T' - T)V$$

$$\text{tg } a = \frac{OO'}{Of} = \frac{v}{V}$$

The effect of this upon the position of a star is easy to see. Consider first a star at the pole of the ecliptic; in this case the earth's motion is always at right angles to the line joining the sun and the earth, and, since the star will be shifted in the direction of the earth's motion, it will describe a small miniature of the earth's orbit, but

always 90° in advance of the earth. Now consider a star in the plane of the ecliptic; since the only component of the earth's motion that can cause a star's place to be shifted by aberration is that across the line of sight, such a star would appear to travel back and forth in a straight line. Obviously, therefore, a star located anywhere in the heavens will describe an aberrational orbit lying between these limits, e. i., a more or less elongated ellipse.

If we were to determine the right ascension and declination of any star tonight, and, after making due allowance for the effect of precession, nutation and aberration, were to compare our results with the co-ordinates of the same star observed 25 or 50 years ago, we would find that the two sets of values would not agree. Further, if we went back another 50 years to a still earlier set of observations and reduced all to a common epoch, we would find a third set of values. The differences between these three sets of values would be proportional to the time, within the errors of the observations. For example, suppose a determination of the right ascension of a certain star had been made in each of the years 1800, 1850 and 1900; we might find, after correcting for the shift of the fundamental planes of reference and reducing all to the common epoch 1800, a set of values somewhat as follows:—

2^h 22^m $17^s.90$ from the observations of 1800.

2^h 22^m $22^s.95$ from the observations of 1850.

2^h 22^m $27^s.93$ from the observations of 1900.

Here it is to be noticed that the right ascension is changing by about $0^s.1$ per year within the limits of error of the observations and in addition to the effects of precession, nutation and aberration. This yearly change is called the star's proper motion.

We may put this in a still more simple way by com-

paring the stars to a lot of chips floating down a stream; all share the common motion of the stream, yet each has an independent motion of its own due to local eddies, currents, and other causes too numerous to mention. Each co-ordinate must have its own proper motion, and the square root of the sum of the squares of the proper motions of two co-ordinates at right angles will give us the actual angular motion of the star across the line of sight. If we multiply this latter by the star's distance from the earth, we will have the actual distance through space that the star travels in the course of a year. The most rapidly moving star known is Groombridge 1830, whose proper motion is 7″ per year. This corresponds to a velocity through space of 138 miles per second. Most proper motions are under 1″ and but few stars have their proper motions determined.

We are now ready to give the methods by which a star's co-ordinates are reduced from one date to another. The American Ephemeris gives a list of 383 stars whose co-ordinates are given for every ten days of the year. This list is sufficient for most purposes of the engineer, but for the more accurate methods of finding latitude the list is not large enough and he must turn to the star catalogues in order to complete his lists.

A star catalogue is simply a list of stars which gives, besides other data which we will discuss later on, what are known as the star's mean co-ordinates for a given year called the date of the catalogue. By a star's mean place we mean its co-ordinates, freed from the effects of nutation and aberration, at the instant of time when the sun's mean longitude is 280°. This always occurs during January 1st; so for all practical purposes we may define a star's mean place for a given year as its co-ordinates on

January 1st of the given year freed from the effects of nutátion and aberration. In reducing the co-ordinates from such a catalogue to their values on the night for which the star's place is desired, we split the problem into two parts as follows: first, compute the effect of precession from the date of the catalogue to the beginning of the year for which the place is required, neglecting entirely the corrections for nutation and aberration; and then, compute the combined effect of precession, nutation and aberration for the remainder of the year. The co-ordinates thus corrected are called the star's apparent place in contradistinction to the co-ordinates for the beginning of the year, which have been corrected for precession alone and are called the star's mean place. The difference between a star's mean and apparent place is called the reduction to apparent place.

This method of procedure is justified by the following considerations. We have seen that precession causes the pole of the ecliptic to describe a circle around the pole of the equator, that nutation causes the pole of the ecliptic to nod back and forth on each side of this circle, and that aberration is a correction which passes through its entire set of values in the course of a year; it is, therefore, immaterial whether we follow the actual wanderings of the pole of the ecliptic through the stars, or travel along its mean path up to the beginning of the year and then make a jump from this mean path at the beginning of the year to its true path at the date required.

1. Correction for Precession.

In giving the formulæ for computing the effect of precession we will consider only the star's right ascension (a), since the formulæ are exactly the same whichever

co-ordinate is desired. From the preceding considerations
we may say that the right ascension varies with the time,
or in other words we may write the equation

(30) $\qquad\qquad a = \psi(T - T_0)$

Here ψ denotes some function of the time $T - T_0$, T_0
being the date of the catalogue and T the year for which
the place is desired. Develop (30) by Maclaurin's Theo-
rem and we have

(31) $\ a = \psi(o) + \dfrac{d\psi(o)}{dT}(T - T_0) + \frac{1}{2}\dfrac{d^2\psi(o)}{dT^2}(T - T_0)^2$

$\psi(o) = a_0$ This is simply the value of the right ascen-
sion for the date of the catalogue and is to
be found in the catalogue.

$\dfrac{d\psi(o)}{dT} - M$ Here M denotes the proper motion and the

entire expression is called the precession.
The numerical value of the precession, to-
gether with that of the proper motion when
known, is given in the catalogue.

$100\dfrac{d^2\psi(o)}{dT^2}$ This is called the secular variation, e. i., the

change in the precession in 100 years.

Substitute these values in equation (31) and we get

$a = a_0 + $ (precession $+ $ M)$(T - T_0) + $
$\qquad\qquad$ 0.005 (secular variation)$(T - T_0)^2$

This may be put into a better form as follows

(32) $a = a_0 + $ [precession $+ $ M $+$
$\qquad\qquad$ 0.005 secular variation$(T - T_0)][T - T_0]$

As an example suppose the mean place of the star
14 Ursæ Majoris was required for the year 1900. From
the Greenwich Ten Year Catalogue for 1880 we take
the following quantities:— In right ascension, $a_0 = 9^{h}$
$01^{m} 0^{s}.535$, precession $= + 5^{s}.0021$, secular variation $=$

$- 0^s.1036$ and the proper motion $= + 0^s.0140$. In declination, $\delta_0 = + 64° 00' 0''.50$, precession $= - 14''.242$, secular variation $= - 0''.507$ and proper motion $= - 0''.067$. We also have $T = 1900$ and $T_0 = 1880$, whence $T - T_0 = 20$ years. The numerical solution is given below.

	a	δ
	h. m. s.	° ′ ″
0.005 sec. var.$(T - T_0)$.	$- 0.0104$	$- 0.051$
precession	$+ 5.0021$ . . .	$- 14.242$
M	$+ 0.0140$	$- 0.067$
Sum	$+ 5.0057$ . . .	$- 14.360$
Sum $\times (T - T_0)$ . . .	$+ 100.114$ . . .	$- 287.2$
Same	$+ 01 40.114$ . .	$- 04 47.2$
a_0 and δ_0	$9 01 00.535$ . .	$64 00 00.5$
Mean a and δ for 1900 .	$9 02 40.65$ . .	$63 55 13.3$

2. Reduction to Apparent Place.

Two methods of computing the reduction to apparent place are in common use, but their derivation is very complex and belongs to a special branch of astronomy. They give rise to two sets of equations containing quantities known respectively as the Besselian Star-Numbers and the Independent Star-Numbers. For field work the Independent Star-Numbers are by far the more convenient and their application to a particular problem is explained below.

The equations containing the Independent Star-Numbers are, in time and arc respectively,

(33) $\quad a = a_0 + f + tM + [g \sin(G + a_0)tg\,\delta_0] \div 15 +$
$\qquad\qquad\qquad [h \sin(H + a_0)\sec\,\delta_0] \div 15$

(34) $\quad \delta = \delta_0 + tM' + g \cos(G + a_0) +$
$\qquad\qquad\qquad h \cos(H + a_0)\sin\delta_0 + i \cos\,\delta_0$

In equations (33) and (34) a_0 and δ_0 are the mean co-ordinates as computed in the preceding section, t is the fraction of a year, and M and M' are the proper motions. The other letters depend upon the position of the sun and moon but are entirely independent of the star's co-ordinates, hence are the same for all stars but vary from night to night. Their values are given in the American Ephemeris for Washington mean midnight of each day of the year. In order, therefore, to compute the reduction from mean to apparent place, it is only necessary to pick out the values of these quantities and substitute them in equations (33) and (34).

In practice the same star will generally be observed on a number of nights, so that more than one apparent place will be desired; in which case the student should not try to interpolate these quantities but should compute the apparent a and δ for Washington mean midnight for every ten days during the period covered by his observations. In this case, f, g and G should be replaced by f', g' and G' given in a supplemental table in the Ephemerides published after 1900; and in case of a single date, f should be replaced by f + f' in that of 1900. Four-place logarithms are sufficient and the + and — signs prefixed to log g, log h and log i as given in the Ephemeris have nothing to do with the logarithms but are used to denote when the number corresponding to the logarithm is either positive or negative. The — sign is equivalent to the subscript n written after a logarithm.

As an example, compute the apparent place of 14 Ursæ Majoris on April 17th, 1900, Washington mean midnight. The constants are found on page 296 of the Ephemeris for 1900 and the solution is given on the next page.

Example of Reduction to Apparent Place.

α_0 in time . 9^h 02^m $40^s.649$ δ_0 . . . . + $63°$ $55'$ $13''.3$

α_0 in arc . . . . $135°$ $40'$ $\cos(G + \alpha_0)$ . . . 9.9204_n

G . . . . $\underline{10 \quad 41}$ log g . . . . . . $\underline{1.0896}$

$G + \alpha_0$ . . . . $146 \quad 21$ Sum . . . . . . 1.0100_n

$\sin(G + \alpha_0)$ . . . 9.7436 G term . . . . — $10''.23$

tg δ_0 . . . . . . 0.3102 $\cos(H + \alpha_0)$ . . . 9.9831

log g . . . . . . 1.0896 log h 1.2823

Sum . . . . . . 1.1434 sin δ_0 . . . . . . $\underline{9.9534}$

log 15 . . . . . . 1.1761 Sum 1.2188

Difference . . . . 9.9673 H term . . . . + $16''.55$

G term . . . . . + $0^s.927$ log i . . . . . . 0.8581_n

α_0 . . . . $135°$ $40'$ $\cos \delta_0$ . . . . . . $\underline{9.6431}$

H . . . . $\underline{240 \quad 12}$ Sum . . . . . . 0.5012_n

$H + \alpha_0$ . . . . . $15 \quad 52$ i term . . . . . — $3''.17$

$\sin(H + \alpha_0)$ . . . 9.4368 t_{M}' . . . . — $0''.02$

log h . . . . . . $\underline{1.2823}$ G term . . . . . — $10''.23$

Sum_1 . . . . . . 0.7191 H term . . . . + $16''.55$

$\cos \delta_0$ . . . . . . 9.6431 δ_0 . . . . $\underline{63° \quad 55' \quad 13''.3}$

log 15 . . . . . . $\underline{1.1761}$ δ . . . . . $63°$ $55'$ $16''.4$

Sum_2 . . . . . . 0.8192

$Sum_1 - Sum_2$ . . 9.8999

H term . . . . + $0^s.794$

f . . . . + $1^s.867$

f' . . . . — $0^s.013$

t_M . . . . + $0^s.005$

G term . . . . . + $0^s.927$

α_0 . . . . . $\underline{9^h \quad 02^m \quad 40^s.649}$

a . . . . . 9^h 02^m $44^s.23$

(1) Find the mean places for 1904 of the following stars from the data given below.

RIGHT ASCENSION FOR 1880

Name	α	Precession	Sec. Var.	P. M.
	h. m. s.	s.	s.	s.
36 Lyncis	9 05 57.036	+ 3.9523	— 0.0375	— 0.0019
φ Bootis	15 33 31.071	+ 2.1479	+ 0.0024	+ 0.0052
Gr. 3150	20 16 20.291	+ 0.5296	— 0.0283	+ 0.0887
52 Ceti	1 38 29.609	+ 2.9065	— 0.0003	— 0.1223
2 Leporis	5 00 22.862	+ 2.5363	+ 0.0033	+ 0.0004

DECLINATION FOR 1880

Name	δ	Precession	Sec. Var.	P. M.
	° ′ ″	″	″	″
36 Lyncis	+ 43 42 41.36	— 14.544	— 0.391	— 0.037
φ Bootis	+ 40 44 41.74	— 11.963	+ 0.257	+ 0.052
Gr. 3150	+ 66 28 04.67	+ 11.238	+ 0.059	+ 0.269
52 Ceti	— 16 34 12.69	+ 18.230	— 0.184	+ 0.857
2 Leporis	— 22 32 01.00	+ 05.158	— 0.360	— 0.068

(2) Compute a ten day ephemeris of the apparent places of the stars given in problem 1, beginning July 1st.

(3) Find the apparent places of the stars given in problem 1 for May 24th, Washington mean midnight.

(4) On July 31st, 1895, at a place whose $\phi = 40°\ 0'\ 0''$, the sun's altitude was found to be $31°\ 01'\ 47''$ at the instant that a sidereal chronometer read $13^h\ 0^m\ 41^s.4$; the barometer read $29''.32$, the attached thermometer read $70°$ F. and the external temperature was $74°$ F. The sun's α and δ were $8^h\ 43^m\ 13^s.6$ and $+ 18°\ 10'\ 52''$. Find the error of the chronometer on local sidereal time.

(5) Suppose that in the above problem, at the same instant of time, the angle between the sun's center and a distant mark was measured and was found to be $123°\ 45'\ 19''$. What would be the azimuth of the mark?

(6) When the planet Mars was at a distance from the earth of six tenths that of the sun, its altitude was found to be 35° 19′ 56″.44 when the barometer was 29″.25, the attached thermometer was 60° F. and the external temperature was 32° F. Find the true altitude of the planet.

(7) Assume that the solar parallax is 8″.85, that the diameter of the earth is 7920 miles, that the length of the year is 365¼ days and that the velocity of light is 186330 miles per second; find the constant of aberration, e. i., the value of the angle a on page 41.

(8) The distance of Mercury is 0.387 that of the sun and its period is 88 days; find the value of the constant of aberration for an observer on Mercury.

(9) Will the precession of the equinoxes ever change the seasons so that winter will come in August?

(10) Will the precession of the equinoxes ever cause the stars which we now see during the winter to be seen during the summer?

(11) From your answers to the two preceding questions, make a definition of the year in common use.

(12) The co-ordinates of a Lyræ are $a = 18^h$ and $\delta = + 39°$; will it ever become the pole star and if so how near will it come to the pole?

(13) Assume that the chronometer in problem (4) was $1^h 34^m 52^s.3$ fast of Washington sidereal time; find the longitude of the place of observation.

(14) At a place whose longitude is $3^h 19^m 49^s.5$ west, the mean time on August 19th, 1904, was $4^h 23^m 18^s.24$ A. M., civil reckoning; find the corresponding sidereal time.

CHAPTER VI.

THE SEXTANT

The sextant belongs to a class of instruments in which the angular distance between two objects, A and B, is measured by superimposing the image of A upon that of B. The ordinary surveyor's transit measures an angle by placing the cross-wires of the telescope upon the image of one object A, as seen in the telescope, then reading the circle and repeating the process for a second object B. The difference of the circle readings gives the angle sought. In the sextant, on the other hand, part of the instrument is turned back and forth until the images of both objects are seen in the field of view at the same time, and then, by means of the slow-motion screw, one 'image is superimposed over the other. The circle reading then gives the angle directly.

Figures VIII and IX, page 52, show a diagrammatic plan of the sextant. A graduated arc whose center is at O carries an arm CE which turns about O. The plane of the graduated arc, GOF, is called the plane of the sextant and the movable arm CE, the index arm. This arm carries a vernier at the end E. Obviously the readings of this vernier give the angle through which the index arm turns. Directly over O and fastened to the index arm is a plane mirror CD, whose face is perpendicular to the plane of the sextant. This mirror is called the index

glass and turns with the index arm. A second mirror, AB, is fixed perpendicularly to the plane of the sextant and always remains in the same position no matter what
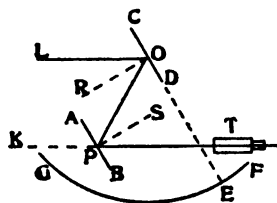
may be the position of the index arm. This mirror is called the horizon glass and only one half of it is silvered, the remainder being left clear, so that a distant object may be seen through it in the direction TK by the eye or a telescope placed at T. The telescope, though not a necessary part of the instrument, increases its efficiency very greatly for astronomical observations and is fastened to the plane of the sextant.

FIG. VIII.

In order to understand the action of this instrument let us suppose the index arm at zero as in Fig. VIII, and the instrument pointed to a distant object such as a star.
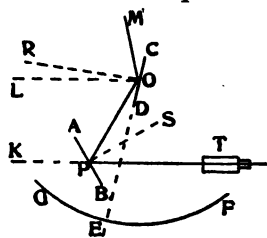
Consider two rays of light from the star; one, KP, strikes the horizon glass AB, passes directly through its unsilvered portion and forms an image of the star in the telescope T; the second ray, LO, strikes the index glass CD whence it is reflected in the direction OP to P, and there suffers a second reflection in the direction PT. It then forms a second image of the object exactly superimposed upon the image formed by the ray KP. It is evident that the slightest rotation of the mirror CD about a vertical axis through O will cause the direction of the ray OP to be changed and hence the direction of the ray reflected from AB; the two images will then no longer coincide.

FIG. IX.

Now suppose the instrument pointed to one star K, and the index arm moved into a new position as in Fig. IX. A ray of light from a second star M will strike the index glass at O, be reflected in the direction OP to the silvered portion of the horizon glass, whence it will be again reflected, and, passing into the telescope T, will form an image of the second star M. Obviously by moving the index E along the graduated circle GF, these two images, one of K and one of M, may be made to coincide exactly. The reading of the graduated circle then gives the angle between them.

To investigate these relations more closely we proceed as follows. Suppose the index arm to be set at zero as in Fig. VIII. The rays LO and KT will then be parallel since the distance OP may be neglected in comparison to the distance of the object observed. Let OR and PS be the normals to the two mirrors CD and AB. Then, since LO and KT are parallel, the angles LOP and OPT are equal. By the law of reflection, OR and PS bisect the angles LOP and OPT. Now suppose the index arm and the mirror CD be rotated until the image of a second object M, formed by reflection from the index glass, coincides with the image of K as seen directly. The condition of the instrument will then be as in Fig. IX. Let R be the reading of the sextant, e. i., the angle through which the index arm and hence the normal, OR, has turned. Let the angle $LOP = 2a$, and $ROL = \delta$.

Obviously then we have in Fig. IX,

$$ROP = ROL + LOP = \delta + 2a$$

and the amount OR has been turned is

(ROP in Fig. IX) — (ROP in Fig. VIII) =

$$\delta + 2a - a = \delta + a$$

Since OR is the normal to the mirror CD, the angle

through which it turns is the same as that through which the index arm turns and is given by the reading of the sextant, R; therefore $R = a + \delta$. LOM is the angle between the two objects. From Fig. IX, we have

$$LOM = LOR + ROM$$

By the law of reflection ROM = ROP, whence

$$LOM = ROP + ROL$$

Substitute the values given above for ROP and ROL and we obtain

$$LOM = \delta + 2a + \delta = 2(a + \delta) = 2R$$

In other words, the angle between the objects is twice the angle through which the index arm turns. To avoid doubling the angle each time the sextant is read, the graduations themselves are numbered double the corresponding angles so that the reading of the instrument gives the angle sought directly.

Adjustments of the Sextant

1. The telescope T should be parallel to the plane of the sextant. Place two pieces of wood of equal thickness at the two extremities of the graduated arc. The thickness of these pieces should be such as to bring their tops on line with the axis of the telescope T. Sight over their tops and make a mark on a distant wall on line and where it can be seen in the telescope. Interchange the pieces of wood and see if the mark is still on line; if not, make a second mark and then a third midway between the first two. Then align the telescope until the middle mark is in the center of the field.

2. The index glass should be perpendicular to the plane of the sextant. This is generally made correct by the maker but should be tested. Put the index arm at the center of the graduated arc, hold the instrument near

the eye, index glass towards you and graduated arc from you, so that half of the graduated arc may be seen at the same time with its image reflected in the index glass. If the reflected half appears the prolongation of the real half, the adjustment may be considered satisfactory; if the two halves seem to bend where they join, the index glass is not perpendicular to the plane of the sextant and should be adjusted.

3. Focus the telescope until the images are sharp.

4. Make both images of the same degree of brightness. For this purpose a screw is generally provided whereby the distance of the telescope from the plane of the sextant may be rapidly varied until the two images are approximately equally bright.

5. The horizon glass should be perpendicular to the plane of the sextant. Point the instrument at a distant mark, preferably the sun or a star, and move the index arm back and forth through the zero point. If the images pass and at no time coincide, the horizon glass is not perpendicular to the plane of the sextant and should be adjusted until one image cuts through the other. This adjustment should be looked to before beginning observing, and, if much out, should be corrected at once.

6. The index error. Set the vernier E at zero and point the instrument at the sun; if the two images coincide the adjustment is satisfactory; if not, they may be made to coincide by rotating the horizon glass about a line perpendicular to the plane of the sextant. It is much better, however, except for rough work, to leave this adjustment considerably out and determine the zero reading, or index error as it is called, before observing. By making this quantity large any mistake of sign is at once detected. To determine this, point the instrument at the

sun and make, say, three pointings, e. i., bring the two images of the sun tangent externally, reading the instrument each time. Do this twice; once with the image seen through the horizon glass above, and once with this same image below. In one of these positions the vernier E will generally be back of the zero point, and, in order to prevent reading the vernier backwards, we count from an imaginary zero 360° back of the true zero. One of the readings will then be 360° + and the other 359° +. Call the first R_1 and the second R_2. Let S be the angular diameter of the sun, I the index error, e. i., the quantity which must be added algebraically to the observed reading in order to obtain the true reading. The first angle measured was obviously the diameter of the sun + 360° and the second was 360° — the diameter of the sun. We have, therefore,

$$R_1 + I = 360° + S$$
$$R_2 + I = 360° - S$$

Solve these two equations for I and we have

(35) $$I = 360° - \tfrac{1}{2}(R_1 + R_2)$$

The sextant is used at sea in finding the ship's position from an observed altitude of the sun. The observer looks through the horizon glass at the line which divides sea and sky, and by moving the index arm "brings the sun down to the horizon", e. i., causes the edge of the sun to touch the sea apparently. It is also frequently used in locating soundings; two angles being measured between three points on shore, the position of the sounding can then be plotted with a three arm protractor. The fact that the sextant requires no rigid mounting but can be held in the hand, adapts it especially for use on moving bodies where a theodolite would be useless. On land we have no true horizon, so it is necessary to use an arti-

ficial horizon. This consists of an iron tray into which is poured a small quantity of mercury, which is then covered with a transparent hood to protect it from the wind. The surface of the mercury being at the same time perfectly level and highly reflecting, a moment's consideration will show that the image of the sun as seen in the mercury will appear just as far below its surface as the real sun appears above. If, therefore, we measure the angle between the real sun and its image in the mercury, we will have twice the sun's altitude.

The greatest drawbacks to the sextant are its limited range and the fact that it is read by a single vernier. The maximum angle the sextant can measure is about 125°; and if the center of the axis of rotation of the index arm does not exactly coincide with the center of the graduated arc, an error will be introduced into the measured angles known as the error of eccentricity. This error is completely eliminated in an instrument having two or more verniers by reading all of them and then taking the mean. For its investigation the student is referred to Doolittle's Practical Astronomy. In any good instrument this quantity is small, and, since its determination is very laborious, for work of such accuracy as to call for it to be taken into account the observer had better use the more accurate methods to be given later.

Hints on Observing

1. Make your observations rapidly. Since the eye soon tires, better results will be secured by moving the tangent screw slowly but steadily in one direction until the images seem in contact and then stopping, than by trying to perfect your setting when once made.

2. In observing the sun always bring the two images

tangent externally and do not try to superimpose them. Make an even number of pointings, half on one edge and half on the other; the mean corresponds to the sun's center.

3. Always keep the sextant slowly rocking about the axis of the telescope T; in this way one image will swing by the other and the contact should be made as the images pass. This is very important.

4. Place the mercury basin well down in the grass, as it forms a cushion to prevent jar and a shield to protect from the wind.

5. If the image seen in the mercury does not appear round, the observer is looking through the wind shield sidewise and by slightly moving his head he may correct the trouble.

6. In finding the sun always get the image reflected from the mercury first. This image remains comparatively still when the sextant is rotated about the axis of the telescope, while the image formed by reflection from the index glass rapidly sweeps across the field of view.

7. After recording your observation, look back at the instrument and make certain the record is correct.

8. In recording an observed time always write down the seconds first, then the minutes, and lastly the hours. You have but one second to get the right second, but sixty seconds to get the right minute.

9. Always date your record. It is better to record the day of the week as well as that of the month.

10. Record the name and number of the instruments used.

11. Never be in a hurry.

CHAPTER VII.

The Theodolite

The engineer's theodolite is diagrammatically shown in Fig. X, page 60. The base, A_1A_2, is provided with either three or four leveling screws E_1, E_2. In this base turns the vertical axis B, to which is fastened the graduated circle R_1R_2, read by the verniers or microscopes V_1, V_2. In large theodolites, reading to single seconds of arc, B and R_1R_2 are permanently fixed together; in small instruments they are connected by a clamp and tangent screw, so that R_1R_2 may be turned about a vertical axis relatively to B. The plate R_1R_2 carries two standards, D_1 and D_2, in which the horizontal axis, F_1F_2, turns. One of the bearings, F_1 or F_2, should be made adjustable so that this axis may be made accurately perpendicular to the axis B. The axis F_1F_2 carries the telescope OG, O being the center of the objective and G the middle cross-wire. To F_1F_2 is fastened the vertical circle H_1H_2. A level, P, is fastened to the arm N, to which the verniers or microscopes for the vertical circle H_1H_2 are fastened. In all theodolites used for astronomical observations, a sensitive striding level, $J_1J_2J_3$, should be provided. This level should rest on the pivots F_1, F_2, and should always be used for the final adjustment in leveling up. The circle R_1R_2 carries two levels, M_1 and M_2, at right angles to each other. By means of these the axis B may be made

FIG. X.

approximately vertical, the adjustment being perfected with the striding level.

There are certain conditions which must be satisfied in a perfect theodolite, and these are seldom met as the instrument comes from the maker. It is the business of the observer to bring his instrument, by means of the adjustments provided, into as nearly the perfect condition as possible; and, where they can not be made perfect, to correct his observations for the outstanding mal-adjustments provided they produce an appreciable error in his work. These conditions and their adjustments are given below.

1. The circle $R_1 R_2$ should be perpendicular to the axis of B, KL.

2. The circle $H_1 H_2$ should be perpendicular to the axis $F_1 F_2$. These conditions are so nearly met in any instrument from a first class maker, that they need not be considered further.

3. The pivots F_1 and F_2 should be round and of the same size. In instruments in which the pivots are less than ¾ inch in diameter this condition is met within our means of measuring the error; in larger instruments, such as meridian circles and astronomical transits, these errors must be taken into account. The method of so doing will be given in the chapter on the transit instrument.

4. The levels M_1 and M_2 should be parallel to the plate $R_1 R_2$; e. i., when the bubbles are at their middle points the plate $R_1 R_2$ should be horizontal and hence the axis B vertical. In order to make this adjustment we proceed as follows: by means of the leveling screws E_1, E_2 bring the bubbles to the middle points of M_1 and M_2; then rotate the instrument 180° about the axis B; if the bubbles are still at the middle points of M_1 and M_2 this ad-

justment is satisfactory; if not, correct half the difference by the leveling screws E_1, E_2 and the remaining half by the adjusting screws of the levels M_1 and M_2 themselves. Repeat this several times until the bubbles remain stationary no matter what may be the position of the instrument. This adjustment should be tested every time the theodolite is set up, and if slightly out may be allowed for by leveling in two positions of the instrument 180° apart, making the bubbles as much out one way in one position as they are the other way in the other position. In all instruments provided with a striding level the final "leveling up" should be done with this, as it is much more sensitive than the so-called plate levels M_1 and M_2.

5. The line OG should be at right angles to the axis F_1F_2. This line is called the line of sight; it is the line joining the center of the objective with the middle cross-wire and gives the direction of any object when the image of the object, formed by the objective, is bisected by the cross-wire. If this adjustment is out, the angle the line OG makes with the true perpendicular to the axis F_1F_2 is called the error of collimation or, more simply, the collimation and is designated by the letter c. This adjustment can seldom be made or kept perfect, and the error of collimation must be taken into account in the measurement of horizontal angles; the methods of doing this will be given under the measurement of horizontal angles. To adjust the collimation level the instrument carefully and point it on a distant mark; then reverse the axis F_1F_2 in the standards D_1 and D_2; if the mark is still bisected this adjustment is perfect, e. i., c = o; if not, correct half the error by means of the adjusting screws provided for that purpose. Should it be impossible to reverse the instrument in the standards D_1 and D_2, the methods for

adjusting the surveyor's transit, given in any text book of civil engineering, may be followed.

6. The level J should be parallel to the horizontal axis F_1F_2, e. i., the two legs should be of the same length and the axis of the level tube should be in the same plane with the axis F_1F_2. For this adjustment bring the bubble to the middle point of J by means of the leveling screws E_1, E_2; reverse the level on the horizontal axis and correct half the error by means of the adjusting screws of the level J, and the other half by means of the leveling screws E_1, E_2. Repeat this several times until the bubble remains at the middle point of J no matter what may be its position on the horizontal axis; the legs will then be of the same length. With a sensitive level it will be found impossible to make this adjustment perfect, but any outstanding error may be completly eliminated by reading the level in two positions on the horizontal axis; and this should always be done. To bring the axis of the level tube into the same plane with the axis F_1F_2, rock the level back and forth on this axis; if the bubble remains stationary the two axes are parallel; if not, move one end of the level tube in azimuth until this adjustment is satisfactory. Any error here may be made insensible by keeping the level J, while it is being read, as nearly as possible in a vertical plane.

7. The axis F_1F_2 should be perpendicular to the axis B. To make this adjustment we proceed as follows:—

(a) In instruments not provided with a striding level. Suspend a plumb bob where it will not be disturbed by the wind, letting the bob be immersed in a bucket of water. Level the theodolite very carefully and point it on the string; then turn the telescope up and down through a vertical angle about the horizontal axis; if the cross-wire

stays on the string, the adjustment is satisfactory; if not, raise or lower one end of the axis $F_1 F_2$ by means of the adjustments provided in one of the uprights D_1 or D_2. This should be repeated in two positions of the instrument which differ by 180° in azimuth.

(b) In instruments provided with a striding level. With all other adjustments made bring the bubble of J to the middle point; then, leaving J on the axis $F_1 F_2$, rotate the theodolite 180° about the vertical axis B; if the bubble remains stationary the adjustment is satisfactory; if not, correct half the error with the adjustment in the support D_1 or D_2. Any error in this adjustment, together with any mal-adjustment of the axis B, unite into one term which should be taken into account in the reduction of the observations. The methods of doing this will be considered under the measurement of horizontal angles.

8. The focusing draw-tube should move in a line parallel to the line OG. Point the instrument on a distant mark and focus carefully, then "set a tack on line" as near the theodolite as possible (e. g., 6 feet) and nearly on a level with the axis $F_1 F_2$; turn the instrument 180° in azimuth and point on the distant mark again; then sight on the tack, focusing carefully; if it is still bisected, the adjustment is correct; if not, the instrument had better go back to the maker, unless the theodolites is provided with an adjustment for this purpose. Wherever the focus remains fixed, as in astronomical observations, this error need not be further considered.

9. Horizontal wires should be horizontal; and vertical wires, vertical. We may assume these two wires are at right angles without error. For this adjustment, place the theodolite so that the horizontal axis is accurately

east and west; then allow a star south of the zenith to
cross the field of view, adjusting the horizontal wire until
the star travels along it. We may also use the plumb bob
as in 7 (a), making the vertical wire coincide with the
string. Any error in the wires may be rendered insensi-
ble by always observing at the same point, e. g., where
they cross.

10. The error of eccentricity: see sextant, page 57.

11. In instruments read by microscopes in place of
verniers, a correction called error of runs should be ap-
plied in order to convert revolutions of the microscope
screw into minutes and seconds of arc. To determine it,
measure with the microscope the interval between two
or more divisions on the graduated circle; the difference
between the angular value as given by the microscope
screw and the value given by the circle, divided by the
number of minutes in the interval covered, gives the
amount to be added to the microscope reading for each
minute of arc. For example: 20′ on the graduated circle
corresponded to 20′ 3″.3 in microscope B; each minute of
arc as given by this microscope must, therefore, be dim-
inished by 0″.16.

Vertical Angles

In measuring vertical angles the bisections should al-
ways be made near the center of the field of view; for if
this be done the errors of collimation (c), and of level of
the horizontal axis (b), may be neglected as producing
no error in the measurements. The center of the field is
usually defined by the intersection of the horizontal and
vertical wires of the eye-piece. It remains, however, to
determine the index error or zero point of the vertical cir-
cle, and the correction due to the alidade level P, Fig. X,

page 60. Fig. XI represents a theodolite as the observer faces the vertical circle, the circle being on his left as he looks in the eye-piece. Fig. XII is the same except that the instrument has been rotated 180° in azimuth and the telescope turned so as to point to the same star. The circle is then on the observer's right as he looks in the
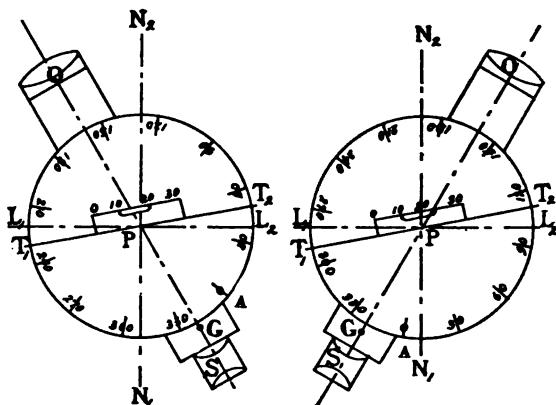
FIG. XI.   FIG. XII.

eye-piece. To understand these two diagrams properly, the student should hold the book vertically in front of him and look at Fig. XI; then turn himself, book and all, 180° in azimuth and look at Fig. XII. By so doing he will see that both figures show the theodolite pointed at the same star, while at first sight he might think that if one shows it pointed south of the zenith, the other would show it pointed north of the zenith; but this is not so.

The following notation is common to both figures:—
OG is the line of sight of the telescope.
$N_1 N_2$ is the true vertical line.

$L_1 L_2$ is the true horizontal line.

$T_1 T_2$ is the line of verniers.

A is the zero point of the graduated circle, the graduations increasing counter-clockwise.

P is the center of the horizontal axis.

Z_0 is the reading of the vertical circle when the telescope points to the zenith and the line $T_1 T_2$ coincides with the line $L_1 L_2$.

L_0 is the reading of the middle point of the bubble when $T_1 T_2$ coincides with $L_1 L_2$, the zero of the level being on the left as the observer faces the vertical circle.

D is the value of one division of the level.

In Fig. XI, let Cl (circle reading, circle left) be the reading of the circle when the telescope is pointed on a star and the condition of the instrument is as shown in Fig. XI.

Let Ll' and Ll" (level, circle left) be the reading of the two ends of the bubble when the circle is on the observer's left as he looks in the eye-piece. The primes always denote the left end of the bubble and the seconds the right end.

Let Zl (zenith distance, circle left) be the true zenith distance of a star when it is bisected by the cross-wire and the circle is on the observer's left as he looks in the eye-piece.

We have from Fig. XI,

$$(36) \qquad Cl = arc\ AT_2 = arc\ AL_2 + arc\ L_2 T_2$$
$$Zl = N_2 PO$$

As theodolites are usually built, the circle is fixed to the horizontal axis and turns with the telescope; hence to point the telescope to the zenith, since the graduations are counter-clockwise, we must increase the arc AL_2 by the angle $N_2 PO$. The resulting angle, arc $AL_2 + N_2 PO$,

will evidently be the reading of the instrument when the telescope points to the zenith and the line $T_1 T_2$ coincides with the line $L_1 L_2$; but we have defined this above as being equal to Z_0, whence

$$Z_0 = \text{arc } AL_2 + N_2PO = \text{arc } AL_2 + Zl$$

Substitute this value of arc AL_2 in (36), solve for Zl and we have

(37) $Zl = Z_0 - Cl + \text{arc } L_2T_2$

To find the value of the arc L_2T_2 we make use of the level readings Ll' and Ll''. The reading of the middle point of the bubble will be $\frac{1}{2}(Ll' + Ll'')$ and, obviously from the figure, this must be greater then L_0, since the high numbered end of the level has been tilted up from the position of perfect adjustment. Whence we have

$$L_2PT_2 = \text{arc } L_2T_2 = D[\frac{1}{2}(Ll' + Ll'') - L_0]$$

where D is the angular value of one division of the level.

Substitute this value of arc L_2T_2 in (37), and put Z_8 for $Z_0 - L_0D$. We will then have

(38) $Zl = Z_8 - [Cl - \frac{1}{2}D(Ll' + Ll'')]$

In formula (38) Cl, Ll', and Ll'' are given directly by observation, D is one of the constants of the instrument; but Z_8 is an unknown constant, called the zenith point, which must be determined for each night's observing. To determine it we proceed as follows: rotate the theodolite 180° in azimuth and point it again on the same star. The condition of affairs will then be as in Fig. XII, page 66. Let Zr (zenith distance, circle right) and Cr (circle reading, circle right) be respectively the true zenith distance and the circle reading in the new position, e. i., circle to the right of the observer as he looks in the eye-piece. We will then have as before

(39) $Cr = \text{arc } AT_2 = \text{arc } AL_2 + \text{arc } L_2T_2$

Obviously from Fig. XII, we must now diminish the

arc AL_2 by N_2PO in order to point the instrument to the zenith. The resulting angle, arc $AL_2 - N_2PO$, will be the reading of the instrument when the telescope points to the zenith and the line $T_1 T_2$ coincides with the line $L_1 L_2$, but this is Z_0; whence we have

(40) $Z_0 = $ arc $AL_2 - N_2PO = $ arc $AL_2 - Zr$

Exactly as before it may be shown that arc $L_2T_2 = D[\frac{1}{2}(Lr' + Lr'') - L_0]$. Substitute the values of the arcs AL_2 and L_2T_2 in (39), put $Z_8 = Z_0 - L_0D$, solve for Zr, and we have

(41) $Zr = Cr - \frac{1}{2}D(Lr' + Lr'') - Z_8$

If we were to make these two observations upon a fixed point on the earth's surface, Zr and Zl would be equal; if, therefore, we subtract equation (38) from equation (41) the left-hand side would become zero. Using the subscript m to denote that the observations are made on the mark, we would then have, after solving for Z_8,

(42) $Z_8 = \frac{1}{2}(Cr_m + Cl_m) - \frac{1}{4}D(Lr'_m + Lr''_m + Ll'_m + Ll''_m)$

Again, if observations of the stars be made rapidly, it may be assumed with sufficient accuracy that the mean of a number of observed altitudes is the same as the altitude corresponding to the average time.[1] Let us then average equations (38) and (41) and for $\frac{1}{2}(Zl + Zr)$ put Z. We then obtain

(43) $Z = \frac{1}{2}(Cr - Cl) + \frac{1}{4}D(Ll' + Ll'' - Lr' - Lr'')$

In equation (43) the quatities within the parentheses represent the averages in each position of the instrument. Thus if three pointings and two level readings had been

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1 This is based upon the assumption that a star's altitude varies proportionately to the time, which is never strictly true and only approximately so for short intervals. For the most refined work a correction, known as a "correction for second differences", should be applied. For the methods of so doing the student is referred to Doolittle's Practical Astronomy.

made in each case, we would have $Cr = \frac{1}{3}(Cr_1 + Cr_2 + Cr_3)$ and $Ll' = \frac{1}{2}(Ll'_1 + Ll'_2)$ and so on; the subscripts representing different observations in the same position of the instrument.

In the preceding demonstration it has been assumed that Cr is greater than Cl, if this is not the case add 360° to Cr. It frequently happens that the zero point of the level is at the middle and the graduations increase in both directions; in this case formulæ (42), (38), (41), and (43) should be replaced by formulæ (44), (45), (46), and (47) given below, which the student should derive for himself. The collected results follow.

Level Graduated from the Left End: Zero on the Left

$$(42)\quad Z_3 = \frac{1}{2}(Cr_m + Cl_m) - \frac{1}{4}D(Lr'_m + Lr''_m + Ll'_m + Ll''_m)$$

$$(38)\quad Zl = Z_3 - [Cl - \frac{1}{2}D(Ll' + Ll'')]$$

$$(41)\quad Zr = Cr - \frac{1}{2}D(Lr' + Lr'') - Z_3$$

$$(43)\quad Z = \frac{1}{2}(Cr - Cl) + \frac{1}{4}D[Ll' + Ll'' - (Lr' + Lr'')]$$

Level Graduated from the Middle

$$(44)\quad Z_3 = \frac{1}{2}(Cr_m + Cl_m) - \frac{1}{4}D(Lr''_m - Lr'_m + Ll''_m - Ll'_m)$$

$$(45)\quad Zl = Z_3 - [Cl - \frac{1}{2}D(Ll'' - Ll')]$$

$$(46)\quad Zr = Cr - \frac{1}{2}D(Lr'' - Lr') - Z_3$$

$$(47)\quad Z = \frac{1}{2}(Cr - Cl) + \frac{1}{4}D[Ll'' - Ll' - (Lr'' - Lr')]$$

Primes, left end of the bubble; seconds, right end.

Note:— Formula (43) or (47) should be used when the observations are to be averaged; and Cr or Cr + 360° should always be greater than Cl. Always take the degrees and minutes from the same vernier or microscope

and the seconds from the mean of the two, recording both readings. Never drop 180° from the vernier adopted to give the degrees and minutes. The method of finding D will be given at the end of the chapter. If the zero of the level is on the right, replace D by — D. Small letters denote the position of the circle.

As an example of the application of the preceding formulæ, we have the following observations of the altitude of Polaris made with the twelve inch theodolite of the Emerson McMillin Observatory:— $D = 2''.0$

Object	C.	Time	Circle Reading	Mic. C		Mic. D		Level	
								R	L
		h. m. s.	° ′	′	″	′	″	d	d
Mark	L		355 45	2	02.0	2	17.2	30.3	10.2
Polaris	L	17 32 50	39 25	0	40.8	0	50.1	33.4	13.2
Polaris	R	17 37 45	140 30	1	24.4	1	28.3	26.3	6.0
Mark	R		184 10	1	19.2	1	31.0	29.2	8.8

REDUCTION

Formula (42)

	° ′ ″
$Cr_m + 360°$.	544 11 25.1
Cl_m	355 47 09.6
Sum — 720° .	179 58 34.7
½ Above . .	89 59 17.4
¼ DΣL	39.2
Diff. = Z_3 . .	89 58 38.2

Formula (38)

Cl	39 25 45.4
½ D(ΣLl)	46.6
Difference . . .	39 24 58.8
Z_3	89 58 38.2
Diff. = Zl . .	50 33 39.4

Formula (41)

	° ′ ″
Cr	140 31 26.4
½ D(ΣLr)	32.3
Difference . .	140 30 54.1
Z_3	89 58 38.2
Diff. = Zr . .	50 32 15.9

Formula (43)

Cr	140 31 26.4
Cl	39 25 45.4
Cr — Cl . . .	101 05 41.0
½ (Cr — Cl) .	50 32 50.5
¼ D(ΣLl — ΣLr) . .	+7.2
Sum = Z . . .	50 32 57.7

In this example the difference between Zr and Zl is due to the star's motion in altitude during the observations. It is to be noticed that the mean of Zr and Zl agrees, within the unavoidable errors of computation, with the value given by (43).

Horizontal Angles

In the measurement of horizontal angles it is necessary to take into account both the errors of collimation, (c), and of the level of the horizontal axis, (b). Consider the triangle of which the verticies are the zenith, (Z), the star, (S), and the point where the prolongation of the circle end of the horizontal axis cuts the celestial sphere, (C).

Let c be the collimation, plus when the telescope points too far in the direction in which the graduations increase, e. i., clock-wise as you look down on the circle.

Let b be the inclination of the horizontal axis, plus when the circle end is too high.

Suppose that the vertical circle is on the left, that c is positive, and b is zero. Since the telescope points too far in the direction in which the graduations increase, if we were to adjust c to zero, we would have to turn the telescope still further in a positive direction to bring the star back on the wire and the new reading would be greater than the old one. It would amount to the same thing, if we were to add to the first circle reading a positive quantity x_1.

Now suppose c to be zero and b to be plus, e. i., the circle end to be too high. If we were to adjust b to zero, the objective end of the telescope would move to the left of the star and would have to be turned to the right, or clock-wise, in order to bring the star back to the wire,

thus increasing the circle reading. In other words we must add a positive correction, x_2, to the original reading in order to obtain the true reading. Obviously, if neither b nor c are zero, we must add a correction, x, to the observed reading such that $x = x_1 + x_2$.

In the triangle ZSC we have, therefore, ZS $=$ Z, the star's zenith distance, CS $= 90° + c$, ZC $= 90° - b$, and the angle SPC $= 90° + x$, since for perfect adjustment this angle would be 90°. Apply formula 2, page 10, to this triangle, using CS for the side a, and we have

$$\cos(90° + c) = \cos Z \cos(90° - b) + $$
$$\sin Z \sin(90° - b) \cos(90° + x)$$

Since b, c, and x are small, we may replace their sines by their arcs and their cosines by unity, and we have after simplifying

$$x = c \operatorname{cosec} Z + b \cot Z$$

Let R1 denote the reading of the horizontal circle when the vertical circle is on the left of the observer as he looks in the eye-piece and R_0 the reading corrected for collimation and level. We will then have

(48) $R_0 = R1 + c \operatorname{cosec} Z + b \cot Z$

Now suppose we reverse the instrument by rotating it 180° in azimuth and then pointing it on the same star again. The vertical circle will then be on the right of the observer as he looks in the eye-piece. Since reversing a telescope always changes the sign of the collimation, we must replace c, in formula (48), by $-$ c for the new position of the instrument. In reversing the theodolite we shall assume that the inclination of the horizontal axis remains unchanged[1], so that now the circle end would be too low and b would be negative. The absolute inclination

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1 In practice this is seldom the case, and a determination of b should always be made in both positions of the instrument.

of the horizontal axis remains unchanged, and hence the level correction must increase the circle reading in the new position exactly as it did in the first position. Since, however, the circle end is now too low, b will be negative, and b cot Z must be subtracted from the circle reading in the new position of the instrument, e. i., circle right. We have moreover increased the circle reading by 180°; whence, if Rr be the circle reading circle right, we have

(49) $R_0 = Rr - 180° - c \operatorname{cosec} Z - b \cot Z$

To determine c we make use of a distant mark on the earth's surface and nearly on a level with the theodolite. With the vertical circle left, point the instrument on the mark and read the horizontal circle calling the reading Rl_m. Reverse the instrument by rotating it 180° in azimuth and pointing it a second time on the mark. The vertical circle will now be on the right and the new reading will be Rr_m. Equations (48) and (49) give us

$$R_0 = Rl_m + c$$
$$R_0 = Rr_m - 180° - c$$

since, Z being 90°, b cot Z is zero. From these two equations we find

(50) $c = \frac{1}{2}(Rr_m - Rl_m) - 90°$

Rr should always be greater than Rl; if it is not, add 360° to Rr. This equation determines c with such a sign that it adds algebraically to readings circle left, and subtracts algebraically from readings circle right.

Again, suppose we measure the angle between two marks on the horizon, both settings being made in the same position of the instrument. Let R_1 and R_2 be the two readings, and R_0 and R_0' the readings corrected for collimation and level; we will then have

$$R_0 = R_1 \pm c$$
$$R_0' = R_2 \pm c$$

The true value of the angle is the difference of the left-hand sides of these equations; but, since c will eliminate in taking this difference, the true value of the angle is the same as the difference of the circle readings. Collimation, therefore, produces no error in the measurement of an angle between two marks on the horizon.

Now suppose we measure the difference in azimuth between a mark on the horizon and a star, once with the instrument circle right, and once with it circle left. Let the two readings on the mark be denoted by the subscript m, and those on the star by the subscript e. Let Z_1 and Z_2 be the zenith distances of the star at the two observations, Ar and Al the true differences of azimuth between the mark and the star, and $A = \frac{1}{2}(Ar + Al)$. From equations (48) and (49), remembering that Ar and Al denote the difference between two pointings, one on the mark and one on the star, we have

(51) $Ar = Rr_m - c - (Rr_e - c \operatorname{cosec} Z_1 - b \cot Z_1)$

(52) $Al = Rl_m + c - (Rl_e + c \operatorname{cosec} Z_2 + b \cot Z_2)$

If the observations are made rapidly, since b is a small quantity, Z_1 may be taken equal to Z_2; whence, averaging the two preceding equations, we have

(53) $A = \frac{1}{2}(Rr_m + Rl_m) - \frac{1}{2}(Rr_e + Rl_e) +$
$\frac{1}{2}(b_1 - b_2) \cot Z$

From formula (51) we see that the collimation eliminates from the mean of two observations in two positions of the instrument, circle right and circle left. A may be taken as the angle corresponding to the average time. [See foot note page 69.]

In order to determine b we make use of the striding level, $J_1J_2J_3$ Fig. X, page 60. First, suppose the zero of the level is at the circle end of the axis as is shown in the upper half of Fig. XIV, page 76, w being the circle end

and E that opposite, the graduations being continuous
from one end. When everything is in perfect adjustment
the bubble will take some position as b_1. Let L_0 be the
reading of the middle of the bub-
ble under this condition. If we
tilt up the circle end of the axis,
the bubble will run to the high
point and take a new position b_2.
Let L_w' and L_w'' be the readings
of the two ends of the bubble in
FIG. XIV.
in this position; $\frac{1}{2}(L_w' + L_w'')$ will then be the reading
of the middle point and will be less than L_0. Formula
(48) was developed upon the supposition that b was plus
when the circle end of the axis was too high, hence we
will have
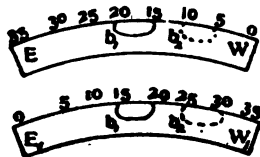
(54) $b = L_0 - \frac{1}{2}(L_w' + L_w'')$

Now suppose we reverse the level on the axis; this
will bring the high numbered end of the level on the cir-
cle end of the axis as shown in the lower half of Fig.
XIV. Let the readings of the ends of the bubble in this
new position be L_e' and L_e''; the reading corresponding
to the middle of the bubble will then be $\frac{1}{2}(L_e' + L_e'')$
and will be greater than L_0, whence we have

(55) $b = \frac{1}{2}(L_e' + L_e'') - L_0$

Lastly, suppose the level itself is out of adjustment,
e. i., one of the legs too long. The effect of this will be
to add an unknown correction, (y), to (54); but, since
reversing the level on the horizontal axis will evidently
change the sign of y, we must subtract y from (55). We
will then have

$$b = L_0 - \frac{1}{2}(L_w' + L_w'') + y$$
$$b = \frac{1}{2}(L_e' + L_e'') - L_0 - y$$

Averaging these two equations and introducing the value

of one division of the level, we have

(56) $b = \frac{1}{4} D[L_e' + L_e'' - (L_w' + L_w'')]$

This equation may be put into a rule as follows: From the sum of the readings when the zero of the level is away from the circle end, subtract the sum of the readings when the zero of the level is at the circle end; then multiply this difference by one fourth the value of one division of the level. Should the zero of the level be at the middle, formula (56) should be replaced by

(57) $b = \frac{1}{4} D[L_1' + L_2' - (L_1'' + L_2'')]$

where the primes denote the circle end of the bubble and the seconds the other end; the subscripts, 1 and 2, refer to the two positions of the level on the horizontal axis.

Collected Results: Horizontal Angles

(50) $c = \frac{1}{2}(Rr_m - Rl_m) - 90°$

(51) $Ar = Rr_m - c - (Rr_e - c \operatorname{cosec} Z - b \cot Z)$

(52) $Al = Rl_m + c - (Rl_e + c \operatorname{cosec} Z + b \cot Z)$

(53) $A = \frac{1}{2}(Rr_m + Rl_m) - \frac{1}{2}(Rr_e + Rl_e) +$
$$\frac{1}{2}(b_1 - b_2) \cot Z$$

b_1 = level, circle right. b_2 = level, circle left.

Subscript m, mark; e, star. Small letters, l and r, denote the position of the circle.

Level: Zero at One End

(56) $b = \frac{1}{4} D[L_e' + L_e'' - (L_w' + L_w'')]$

Subscript w, zero of level at circle end; subscript e, zero of level at opposite end.

Level: Zero at the Middle

(57) $b = \frac{1}{4} D[L_1' + L_2' - (L_1'' + L_2'')]$

Primes, circle end of bubble; seconds, opposite end

Note:— In any position of the instrument, readings on the star must always be greater than readings on the mark in the same position of the instrument; if not, add 360°. In formula (50), Rr_m should be greater than Rl_m,

if not, add 360°. In recording level readings, always fol-
low the form given below, where R denotes the right end
and L the left end of the bubble as the observer looks in
the eye-piece. These formulæ always lead to a negative
angle; hence, it is only necessary to add it algebraically
to the computed azimuth of a star in order to obtain the
azimuth of the mark.

As an example of the application of the preceding
formulæ, we have the following observations of the dif-
ference of azimuth between Polaris and a fixed mark on
the earth's surface made with the twelve inch theodolite
of the Emerson McMillin Observatory. The zero of the
level was at the middle. D = 2″.5.

Object	C.	Time	Circle Reading	Mic. A	Mic. B	Level R L
		h. m. s.	° ′	′ ″	′ ″	d d
Mark	R		330 30	1 55.1	1 57.4	26.0 34.0
Polaris	R	14 04 57	74 05	2 00.3	2 02.1	27.0 35.5
Polaris	L	14 11 50	254 05	4 19.1	4 17.2	35.0 27.0
Mark	L		150 30	1 25.0	1 21.5	34.0 28.0

The error of runs for both microscopes equals — 0″.17
multiplied by the reading of the microscope in minutes.
$Z_1 = Z_2 = 51°$; cot Z = 1.24, cosec Z = 1.29.

REDUCTION

Formula (57)   Formula (57) Cont'd
Circle Right   Circle Left

$\Sigma L'$ . . . . 53.0 $\Sigma L'$ 55.0
$\Sigma L''$ . . . . 69.5 $\Sigma L''$ 69.0
$\Sigma L' - \Sigma L''$ . . . — 16.5 $\Sigma L' - \Sigma L''$ . . . — 14.0
b_1 . . . . — 10″.3 b_2 . . . . — 8″.8
b_1 cot Z . . . . . — 12″.9 b_2 cot Z . . . . . — 11″.0

Formula (50)				Formula (53)			

$$Rr_m \ . \ . \ . \ . \ . \ 330 \ 31 \ 55.9 \quad Rr_m \ . \ . \ . \ . \ . \ 330 \ 31 \ 55.9$$

$$Rl_m \ . \ . \ . \ . \ . \ 150 \ 31 \ 23.0 \quad Rl_m \ . \ . \ . \ . \ . \ 150 \ 31 \ 23.0$$

Difference . . 180 00 32.9   Sum 481 03 18.9

c . . . . + 16.4   ½ Sum 240 31 39.4

c cosec Z . . . . + 21.3 Rr_e 434 07 00.9

Formula (51) Rl_e 254 09 17.5

Rr_e . . . . . 434 07 00.9   Sum 688 16 18.4

— c cosec Z . . . — 21.3   ½ Sum 344 08 09.2

— b cot Z . . . . + 12.9   Difference . — 103 36 29.8

Sum . . . . . 434 06 52.5 b_1 cot Z . . . . . — 12.9

Rr_m — c . . . 330 31 39.5 b_2 cot Z . . . . . — 11.0

Ar . . . . — 103 35 13.0   ½ Difference . . . — 0.9

Formula (52)   A . . . . — 103 36 30.7

Rl_e . . . . . 254 09 17.5   The difference between
c cosec Z . . . . + 21.3   Ar and Al is caused by the
b_2 cot Z . . . . . — 11.0   star's motion in azimuth be-
Sum . . . . . 254 09 27.8   tween the two observations.
Rl_m + c . . . 150 31 39.4
Al . . . . — 103 37 48.4

Value of one Division of the Level: D

To determine the value of one division of the level we will follow a method due to Comstock and given by him in his text book "Field Astronomy for Engineers". After leveling the theodolite carefully, place the horizontal axis approximately parallel to two of the leveling screws. Then by moving the other screw or screws, throw the vertical axis considerably out of the vertical, and measure the angle through which it has been moved. This can easily be done, if the instrument has a vertical circle, by pointing on a distant mark when the theodolite is level

and reading the vertical circle; then, after throwing it
out of level, pointing the telescope on the same mark and
reading the circle a second time. Should the instrument
not have a vertical circle, we may use a graduated rod
held vertical at a known distance and on a level with the
instrument. Evidently two readings on the rod, com-
bined with its known distance, will enable us to calculate
the angle through which the vertical axis of the theodo-
lite has been turned.

There will be two positions of the theodolite, $180°$
apart, at which the bubble of the striding level will be at
its middle point. Call the reading of the horizontal circle
at either of these points R_0. It will be found that by
slowly rotating the theodolite in azimuth the bubble may
be made to move over any desired number of divisions.
Place the axis so that the bubble will be near one end of
the level tube and call the reading of the horizontal circle
R_1. The horizontal angle through which the theodolite
has turned is then $R_1 - R_0$; and the value of one division
of the level is given by the equation

$$D = [e \sin (R_1 - R_0)] \div N$$

where N is the number of level divisions the bubble has
passed over while the theodolite moved from the position
R_0 to the position R_1, and e is the inclination of the ver-
tical axis to the true vertical.

In order to prove this equation, let us consider the
spherical triangle of which the vertices are the points at
which the circle end of the axis prolonged cuts the celes-
tial sphere in the two positions R_0 and R_1, and the point
at which the vertical circle through the second of these
two points cuts the horizon. Call these three points A,
B, and C; and we will have $BAC = e$, $AB = R_1 - R_0$,
$BC = b$ the elevation of the circle end of the axis, and

the angle BCA $= 90°$. We then have
$$\sin b = \sin e \sin(R_1 - R_0)$$
The value of D will then be equal to b divided by the number of divisions the bubble has passed over; furthermore, b and e are small, so that we may write
$$D = [e \sin(R_1 - R_0)] \div N$$
In practice it is better not to determine R_0 but to run the bubble to one end of the scale and call the reading R_1, then to the other end of the scale and call the reading R_2. We may then compute D by the formula
$$(58) \qquad D = [e \sin(R_1 - R_2)] \div N$$
As an example of the above method we have the following observations of one division of the striding level of a five inch theodolite. The instrument was leveled, pointed on a distant mark, and the vertical circle read $4° 8' 20''$; it was then thrown out of level and pointed on the same mark again, when the vertical circle read $3° 41' 30''$; e, therefore, equals $1590''$. The additional observations and their reductions follow:

R		Difference		Level R L		Half of the Difference	N
o	′	o	′	d	d	d	d
42 02		·4	57	1.2	13.8	+ 6.3	12.8
46 59		4	55	14.0	1.0	— 6.5	13.0
42 04		4	10	1.0	14.0	+ 6.5	11.0
46 14		4	01	12.0	3.0	— 4.5	10.5
42 13		4	29	1.5	13.5	+ 6.0	11.5
46 42				13.0	2.0	— 5.5	

The first column gives the readings of the horizontal circle when the bubble had the position given in column three. The second column gives the differences without regard to sign, e. i., $R_1 - R_2$. The fourth column gives

half the difference of the level readings. The last column ·gives the differences of the quantities in column four. The logarithmic computation is given below.

$\sin(R_1 - R_2)$ . . . 8.9359 8.9330 8.8613 8.8454 8.8930
$e \sin(R_1 - R_2)$ . . 2.1373 2.1344 2.0627 2.0468 2.0944
$\log N$ 1.1072 1.1139 1.0414 1.0212 1.0607
$\log D$ 1.0301 1.0205 1.0213 1.0256 1.0337
D . . . . 10″.7 10″.5 10″.5 10″.6 10″.8

Hints on Observing

1. Test all adjustments and if much out adjust.

2. Focus carefully. The eye-piece should send out parallel rays, e. i., when the observer looks into it his eye should be at rest as when looking at a distant object. This can be easily done by racking the objective out of focus and pointing the telescope at the sky near the horizon; then, keeping both eyes open, with one look directly at a distant mark and with the other look in the eye-piece; then focus the eye-piece until the cross-wires are seen distinctly with one eye, while at the same time the distant mark is seen distinctly with the other. The telescope may then be pointed on the mark or star and the objective focused.

3. Unless the star moves very slowly, do not try to bisect it, but place the wire near the star and observe the time of transit.

4. Never touch the pier which carries the instrument.

5. Always use two hands to turn the instrument in azimuth.

6. At night, when reading a level, hold the lamp as far from the level as possible.

CHAPTER VIII.

AZIMUTH

All extensive surveys rest primarily upon three quantities which must be determined by observations on the stars. These quantities are time, latitude, and azimuth. The astronomer has devised numerous methods of determining each of these quantities; some of which, being more or less crude, are adapted for rough reconnaissance work with small portable instruments; others, reaching the limit of mathematical analysis, are used only with instruments of great power and in the most refined work of modern geodetic surveys.

These several methods may be grouped into three divisions as follows: (1) those depending upon the measurement of horizontal angles; (2) those depending upon the measurement of vertical angles; and (3) those methods which demand an instrument of a special form of construction. Under (1) we find all the methods of measuring azimuth, under (2) the cruder, and under (3) the more refined methods of measuring time and latitude.

In this chapter we shall consider only two of the numerous methods of finding azimuth. They all rest upon the fundamental principle that, knowing the declination of a heavenly body, the latitude of the observer, and the instant of time at which the observation was made, we can compute the azimuth of the body at the instant of

observation, and hence the azimuth of the mark, provided that at the instant we made the observation we also measured the difference of azimuth between the mark and the star.

A glance at Fig. I, page 6, shows us that the process consists simply in the solution of the triangle PZS for the angle at Z, where we have given $PZ = 90° — \phi$, $PS = 90° — \delta$, and the angle $ZPS = t$, the hour angle of the star. The nearer a star is to the pole the slower its apparent motion in azimuth, and the less will any error of the observed time of bisecting it affect the result. It is therefore evident that for the most refined work a star close to the pole should be selected; Polaris, the north star, is especially suited. With such stars, however, the side PS of the triangle becomes very short and the general methods of Chapter II should be replaced by a formula to be given later. For rough work the sun may be used, provided we are satisfied with an accuracy of about one minute of arc. In observing the sun for azimuth the observations should always be made when the sun is near the prime vertical, e. i., the vertical circle through the east and west points of the horizon.

Azimuth from the Sun

Observing Program

1. Level the theodolite and test all adjustments.

2. Point on the mark, circle right, reading both verniers.

3. Make three pointings on one limb of the sun, circle right, reading both verniers each time and recording the times at which the bisections were made. Read striding level at the beginning of the set, reverse, and read it again at the end of the set.

4. Repeat (3), circle left, bisecting the opposite limb of the sun.

5. Repeat (2), circle left.

Process of Reduction

1. Compute the difference of azimuth between the mark and the sun by the formulæ and examples on pages 77, 78, and 79. If formula 53 is used, average the times.

2. Compute the Washington mean time by the formula $T + \Delta T + L$, where ΔT is the chronometer error on local mean time and L is the longitude west of Washington.

3. Compute the value of δ as follows: pick out of the American Ephemeris δ_0, the declination for Washington mean noon, and the hourly variation in declination; multiply the hourly variation in declination by $T + \Delta T + L$ expressed in hours and add the result, $(\Delta\delta)$, to δ_0.

4. Find the local apparent time by the rules on page 30. This gives the sun's hour angle.

5. Compute the azimuth of the sun by the formulæ of example 2, page 17.

6. Add (1) and (5).

Note:— If the observations are reduced separately the resulting azimuths should be corrected for the fact that the observations were made on the edge or limb of the sun and not on the center. A moment's consideration will show that the error thus introduced is exactly the same as that which would be introduced if we were to set upon the center of the sun with a cross-wire whose error of collimation is equal to the sun's semi-diameter. In order, therefore, to correct for this, it is only necessary to add ± S cosec Z to the computed azimuth of the sun, S being its semi-diameter as taken from the American Ephemeris. If a star had been observed this need

not be considered as the star has no appreciable diameter. If a star near the prime vertical had been observed in place of the sun (2) and (3) should be omitted, and in their place we should pick out a and δ from the Ephemeris for the nearest day. The times, if given in mean time, should be converted into sidereal times by the rules on pages 29 and 30, and the star's hour angle found by subtracting its right ascension from the true sidereal time.

As an example of the above, we take the following observations of the sun for azimuth, made with a five inch theodolite of the Emerson McMillin Observatory.

DATE:— September 6th, 1904.

Object	C.	Time	Vernier A	Ver. B	Level R L
		h. m. s.	o ′ ″	′ ″	d. d.
Mark	R		0 0 0	02 00	7.5 7.8
Sun	L	20 00 14.2	35 51 30	53 50	7.0 8.0
Sun	R	20 11 51.5	218 55 20	56 40	7.5 7.5
Mark	L		180 00 40	02 20	6.0 8.5

From the American Ephemeris for 1904, page 405, we find the declination for Washington mean noon (δ_0) $= + 6° 26′ 02″.4$, the hourly change in declination $= - 55″.97$, $E_0 = - 1^m 42^s.5$, $S = 15′ 54″$, and the change in E in $24^h = - 19^s.9$. The known constants of the observatory and chronometer are $\phi = 39° 59′ 50″$, $L = 23^m 46^s.8$ west, $\Delta T = + 28^m 34^s.6$, and $\Delta T + L = 52^m 21^s.4$.

NOTES

* This computation is not given here; the student should, however, provide for it in making out his own form for reduction.

† The 24^h is subtracted because the observations were

Reduction: Azimuth from the Sun

	Circle Left	Circle Right
*A, see pages 77 to 79.	$-215°\ 51'\ 09''$	$-218°\ 55'\ 07''$
Observed time $=$ T	$20^h\ 00^m\ 14^s.2$	$20^h\ 11^m\ 51^s.5$
T $+$ ΔT $+$ L	$20\ \ 52\ \ 35\ .6$	$21\ \ 04\ \ 12\ .9$
†T $+$ ΔT $+$ L $-$ 24^h	$-3\ \ 07\ \ 24\ .4$	$-2\ \ 55\ \ 47\ .1$
Same in hours	$-3^h.120$	$-2^h.930$
Δδ	$+2'\ 54''.6$	$+2'\ 44''.0$
δ	$+6°\ 28''\ 57''$	$+6°\ 28'\ 46''$
T $+$ ΔT $+$ L $-$ E₀	$20^h\ 54^m$	$21^h\ 06^m$
†Same $-$ 24^h, in days	-0.129	-0.121
ΔE	$+2^s.6$	$+2^s.4$
E $=$ E₀ $+$ ΔE	$-1^m\ 39^s.9$	$-1^m\ 40^s.1$
T $+$ ΔT	$20^h\ 28^m\ 48^s.8$	$20^h\ 40^m\ 26^s.1$
T $+$ ΔT $-$ E	$20\ \ 30\ \ 28\ .7$	$20\ \ 42\ \ 06\ .2$
Sun's hour angle	$307°\ 37'\ 10''$	$310°\ 31'\ 33''$
cot δ	0.94453	0.94473
cos t	9.78563	9.81277
Sum $=$ cot Z	0.73016	0.75750
Z	$10°\ 32'\ 40''$	$9°\ 54'\ 52''$
$\phi - $ Z	$29\ \ 27\ \ 10$	$30\ \ 04\ \ 58$
tg t	$0.11315n$	$0.06811n$
cos Z	9.99261	9.99346
Sum	$0.10576n$	$0.06157n$
Sin($\phi - $ Z)	9.69171	9.70006
Diff. $=$ tg A	$0.41405n$	$0.36151n$
‖A $=$ sun's azimuth	$291°\ 04'\ 42''$	$293°\ 30'\ 32''$
cot($\phi - $ Z)	0.24819	0.23711
cos A	9.55587	9.60085
Sum $=$ cot z	9.80406	9.83796
z	$57°\ 30'$	$55°\ 27'$
$\pm$ S cosec z	$18'\ 51''$	$18'\ 59''$
‡Azimuth of the mark	$74°\ 54'\ 42''$	$74°\ 54'\ 23''$

made in the morning, the date is in civil reckoning, and the interpolation should always be made from the nearest date given in the tables.

|| The quadrant is most easily determined as follows: the observations being made in the morning, the sun's azimuth must lie either in the third or fourth quadrant, and the negative tangent shows that it is the fourth.

‡ Azimuth mark = azimuth sun + A ± S cosec z.

The level readings given in the problem were all taken on the sun and none on the mark as might seem from the record of the observations. $D = 10''$.

If the method of averages had been used, we would compute A by formula (53), which gives — $217° 55' 07''$; for T we would use $\frac{1}{2}(20^h 00^m 14^s.2 + 20^h 11^m 51^s.5) =$ $20^h 06^m 02^s.8$; then, exactly as before, using this value of T we find the azimuth of the sun = $292° 16' 53''$, and the azimuth of the mark = $74° 53' 45''$, the sun's semi-diameter eliminating. It will be noticed that this result does not agree with the mean of the two reduced separately; this is due to the neglected second differences.[1] It should be stated that in observations made on Polaris this error is not nearly so great, but, nevertheless, for the best work this correction should be applied or the observations reduced separately. It is shorter to apply the correction, but the extra labor of reducing them separately is not great and, unless one has much of this work to do, it hardly pays to use second differences, the separate reductions being equally accurate and in some ways better.

Azimuth from Polaris
Observing Program

The observing program is the same as in azimuth from the sun. It should be stated that the number of pointings

1 See foot note page 69.

to be made in each case is a matter of judgment for the observer; there should be, however, an equal number in each position of the instrument.

Process of Reduction

1. Same as in azimuth from the sun.

2. Pick out a and δ for the nearest day, except for the most refined work, where they should be interpolated for the nearest tenth of a day.

3. If times are not already given in sidereal time convert them, and compute the hour angle of Polaris by the formula $t = T + \Delta T - a$, where T is the observed sidereal time and ΔT the clock error.

4. Compute the azimuth of Polaris, and, for refined work, the correction for diurnal aberration by the methods given below.

5. Add (1) and (4).

To find the azimuth of Polaris we will follow the demonstration given in Doolittle's "Practical Astronomy".

Apply formula (1), page 10, to the triangle ZPS, Fig. I, page 6, and we have, a being the azimuth from the north point and $p = 90° - \delta$,

(59) $\sin a \sin Z = - \sin p \sin t$[1]

Apply formula (5), page 11, to this same triangle, calling A of the formula the angle at Z and B that at P; we will then have, after simplifying,

(60) $\sin Z \cos a = \cos p \cos \phi - \sin p \sin \phi \cos t$

Divide (59) by (60), reduce, and we have

(61) $tg\ a = - \dfrac{\sin p \sin t}{\cos p \cos \phi - \sin p \sin \phi \cos t}$

From the calculus, since a and p are small, we have

---

1 Hour angles to the west are positive and to the east negative; therefore, with a positive hour angle the star would be west of north or in a direction from the north opposite to that in which the graduations of the circle increase, and hence should be counted negative.

$\sin p = p — \frac{1}{6}p^3$, $\cos p = 1 — \frac{1}{2}p^2$, and $\tan a = a + \frac{1}{3}a^3$, neglecting powers higher than the third. Substitute these values in (61) and reduce, neglecting terms higher than the third, and we obtain

(62) $a \cos \phi = — p \sin t + a p \sin \phi \cos t +$
          $\frac{1}{2} a p^2 \cos \phi — \frac{1}{3} a^3 \cos \phi + \frac{1}{6} p^3 \sin t$

In equation (62) the unknown quantity a is found on the right-hand side but only in terms of the second and third order, therefore we may solve the equation by a method of approximation. First, neglect terms higher than the first and we have

$a \cos \phi = — p \sin t$ or $a = — p \sec \phi \sin t$

Substitute this in (62), neglect terms higher than the second, and we have after reduction

$a = — \sin t \sec \phi (p + p^2 \tan \phi \cos t)$

Substitute this value in the second, third, and fourth terms of (62), introduce the $\sin 1''$ to convert from radians to arc, reduce, and we have finally

(63) $a = — \sec \phi \sin t [p + p^2 \sin 1'' \tan \phi \cos t +$
        $\frac{1}{3} p^3 \sin^2 1'' ([1 + 4 \tan^2 \phi] \cos^2 t — \tan^2 \phi)]$

For Polaris, within the limits of the United States, the last term will be less than $2''$ while those neglected will be less than $0''.1$. In the second term $p^2 \sin 1'' \tan \phi$; in the fourth term $\frac{1}{3} p^3 \sin^2 1''$, $1 + 4 \tan^2 \phi$, and $\tan^2 \phi$ are the same for all observations made on the same night and should be computed once for all. All terms except the first may be computed with four place tables. Formula (63) need only be used with theodolites reading to single seconds; with those reading to ten seconds use

(64) $a = — \sin t \sec \phi [p + p^2 \sin 1'' \tan \phi \cos t]$

The azimuth of Polaris from the south is then given by

(95) $A = 180° + a$

We have seen in Chapter V, that the earth's motion

combined with the velocity of light causes the star to be displaced in the direction of the earth's motion; now the rotation of the earth on its axis carries the observer with it and hence causes the star's position to be shifted. This effect is very small, much smaller than that given in Chapter V which was due to the earth's orbital motion about the sun. Its effect on the star's azimuth is given by

(66) $\Delta a = 0''.319 \cos \phi \cos a \sec h$

For a close circum-polar star this may be simplified into

(67) $\Delta a = 0''.319 \cos a$[1]

Δa should always be added algebraically to the computed azimuth of the star and obviously need be taken into account only in the most refined work.

The following observations for azimuth, made with the twelve inch theodolite of the Emerson McMillin Observatory, will illustrate this method.

<p align="center">DATE:— August 27th, 1904.</p>

Object	C	Time	Circle Reading	Mic. A	Mic. B	Level Direct R L	Level Reversed R L
		h. m. s.	° ′	′ ″	′ ″	d. d.	d. d.
Mark	R		330 30	2 01.7	1 59.0	37 27	41 24
Polaris	R	18 58 18	75 20	4 59.5	4 56.3	32 32	34 30
Polaris	L	19 09 40	255 20	4 42.9	4 35.1	30 34	33 31
Mark	L		150 30	1 29.8	1 25.8	40 24	38 27

In the above example the zenith distance of the mark was 94°; the theodolite read directly to single seconds of arc, hence it was necessary to take into account the level of the horizontal axis when the theodolite was pointed on the mark. This is easily done by replacing in the formulæ on page 77, Rr_m and Rl_m by $Rr_m - br_m \cot Z_m$ and $Rl_m + bl_m \cot Z_m$ respectively, and in formulæ (51) and

⸱   1 See Doolittle's "Practical Astronomy" pages 530, 531, and 532 for the proof of these formulæ.

Reduction: Azimuth from Polaris

	Circle Right	Circle Left
Ar or Al	$- 104° 52' 47''.6$	$- 104° 53' 13''.9$
Observed T . . .	$18^h 58_m 18^s.0$	$19^h 09^m 40^s.0$
T + ΔT — a . .	$17^h 33^m 19^s.6$	$17^h 44^m 41^s.6$
t in arc	$263° 19' 54''$	$266° 10' 24''$
cos t	$9.0649n$	$8.8244n$
log p^2 sin $1''$ ×		
tg ϕ cos t	$0.9501n$	$0.7096n$
p^2 sin $1''$ tgϕ cos t	$- 8''.9$	$- 5''.1$
log cos² t	8.1298	7.6488
log (1 +		
4 tg² ϕ) cot² t	8.7114	8.2304
(1 + 4 tg² ϕ) cos² t	$+ 0.052$	$+ 0.017$
(1 + 4 tg² ϕ) cos² t		
— tg² ϕ	$- 0.652$	$- 0.687$
log of the above	$9.8142n$	$9.8370n$
Sum above and		
log ⅓p^3 sin² $1''$	$9.6220n$	$9.6448n$
Number	$- 0''.4$	$- 0''.4$
p^2 sin $1''$ tg ϕ cos t	$- 8''.9$	$- 5''.1$
Sum	$- 9''.3$	$- 5''.5$
Above + p . . .	$4335''.1$	$4338''.9$
log of the above .	3.63700	3.63738
sin t	$9.99705n$	$9.99903n$
Sum	$3.63405n$	$3.63641n$
Sum — log cos ϕ	$3.74978n$	$3.75214n$
a in $''$	$5620''.5$	$5651''.1$
a in $°$ $'$ $''$ . . .	$1° 33' 40''.5$	$1° 34' 11''.1$
180° + Δa + a .	$181° 33' 40''.8$	$181° 34' 11''.4$
Ar or Al	$- 104° 52' 47''.6$	$- 104° 53' 13''.9$
Azimuth mark .	$76° 40' 53''.2$	$76° 40' 57''.5$

(52), c cosec Z by c sin Z_m cosec Z and then use the formulæ as they stand. In this problem, however, c sin Z_m differs by only $0''.1$ from c and we may neglect this last correction. It should be noted that in this problem, Z_m being 94°, cot Z_m will be negative. From the Ephemeris for 1904, page 319, we find $a = 1^h 25^m 37^s.0$, $\delta = 88° 47' 35''.6$; we also have given $\phi = 39° 59' 50''$, ΔT on local sidereal time $= + 38^s.6$, and $L = 23^m 46^s.8$ west. From the above data we find $p = 4344''.4$, $\log p^2 \sin 1'' tg \phi = 1.8852$, $\log \frac{1}{3}p^3 \sin^2 1'' = 9.8078$, $\log(1 + 4 tg^2 \phi) = 0.5816$, $tg^2 \phi = 0.704$, and $a - \Delta T = 1^h 24^m 58^s.4$. The error of runs is the microscope reading in minutes multiplied by $- 1''.05$ for A, and by $- 0''.15$ for B. D is $2''.5$. In accordance with the directions given above we find $c = + 16''.0$, c cosec $Z = + 20''.9$, $Rr_m - br_m$ cot $Z_m - c = 330° 31' 44''.3$, and $Rl_m + b_m$ cot $Z_m + c = 150° 31' 44''.2$. The level reads from the middle, $Z = 50°$, $\Delta a = 0''.3$, and cos $\phi = 9.88427$. The rest of the computation is given on the preceding page.

If the method of averages had been used we would compute A by formula (53), modified as above to include the level readings on the mark. This gives $A = - 104° 53' 0''.8$. For T we would use $\frac{1}{2}(18^h 58^m 18^s + 19^h 09^m 40^s) = 19^h 03^m 59^s$. Exactly as before, we would find from this value of T, $a = 1° 33' 56''.9$ and the azimuth of the mark $= 181° 33' 57''.4 + 0''.3 - 104° 53' 0''.8 = 76° 40' 56''.9$. This result differs by under two seconds from the mean of the two reduced separately; this error is due to the neglected second differences, and is about the same as the accidental error of observation.

CHAPTER IX.

TIME

In this chapter we shall take up the cruder methods of finding time from observations of the heavenly bodies, reserving for the chapter on the transit instrument the discussion of a more refined process. The methods here considered all depend upon measuring the altitude of the sun or a star and in addition noting, on the clock or chronometer whose error is to be found, the exact instant at which the observation was made. If we look at Fig. I, page 6, we will see that in the triangle PZS our measured altitude, when corrected for refraction and parallax, gives us the side $ZS = 90° — h$, the known latitude gives us the side $PZ = 90° — \phi$, and from the Ephemeris we can pick out the star's declination which gives us the side $PS = 90° — \delta$. We have given, therefore, the three sides of the triangle and hence can compute the hour angle by formulæ (20) and (21), page 14, and with the hour angle known the true time by the rules on pages 29, 30, and 31.

There is a second and more accurate method of finding time which is well suited for work in the field. This consists of measuring several altitudes of a star when it is east of the meridian, noting the corresponding times, and again observing the times when the star has the same series of altitudes west of the meridian. Turning again to Fig. I, we see that we will then have two triangles, one

east and one west of the meridian, in which all the sides
are the same, the star's declination not changing appre-
ciably between the two observations. Since the sides are
the same the hour angles must be also, and, if T_e and
T_w be the observed times of the eastern and western ob-
servations, t_e and t_w the corresponding hour angles, ΔT
the clock correction, and a the star's right ascension, we
will have

$$T_e + \Delta T = a - t_e \text{ and } T_w + \Delta T = a + t_w$$

We have seen above that $t_e = t_w$, whence we have

$$\Delta T = a - \tfrac{1}{2}(T_e + T_w)$$

In the case of the sun the formula becomes more com-
plex because of the relatively rapid change in the sun's
declination which must be taken into account; the method
of so doing will be given later.

Time by a Single Altitude

Observing Program: Theodolite

If a theodolite is used the observing program given
under azimuth from the sun should be followed except
that the bisections should be made with the horizontal
cross-wire upon the upper and lower limbs of the sun,
the vertical circle should be read, and the alidade level
in place of the striding level. It is well to read this level
after each observation as it is apt to change rapidly. In
addition to this the barometer, attached thermometer,
and external temperature should be read either at the be-
ginning or end of the set.

Observing Program: Sextant

1. Test all adjustments.

2. Observe for index error as explained under 6, pages
55 and 56. At least three pointings should be made in
each position, and this had better be done both at the be-
ginning and end of each day's work as the index error is

liable to change during the observations.

3. Fill the artificial horizon with mercury and direct the long way of the trough towards the sun. If a glass wind shield is used reverse this at the middle of the set.

4. Set the sextant at some even 10′ mark slightly in advance of the sun, e. i., so that the two images are approaching; then observe five contacts, setting the sextant 10′ ahead each time and recording the times.

5. Go back to the first reading of (4) and it will be found that the two images overlap but are now separating and we may observe five contacts over the same set of readings as in (4). Obviously the average altitude of the sun's center is that which corresponds to the middle sextant reading and may be considered as corresponding to the average time.

6. Read barometer and thermometers as above.

Process of Reduction

1. Compute the observed altitude: for a theodolite, by the formulæ on page 70; for a sextant, by computing the index error by formula (37), page 56, adding this to the sextant reading, and dividing the result by two.

2. Correct for refraction and parallax by the methods of Chapter III and, if the observations are reduced separately, add or subtract the sun's semi-diameter taken from the Ephemeris.

3. Compute the value of δ exactly as in azimuth from the sun or a star near the prime vertical. If the sun has been observed it is necessary to know an approximate value of ΔT, but if the assumed value is one minute in error the resulting value of δ will be less than one second of arc in error.

4. Compute the hour angle by formulæ (20) and (21), page 14.

5. If the sun is observed, (4) gives us the apparent solar time which should be converted to the kind of time kept by the chronometer by the rules on pages 29, 30, and 31. If a star is observed, the true sidereal time is given by $T = a + t$.

6. Subtract the observed time from the true time and the result will be ΔT.

As an example of the above we have the following observations of the sun made with a sextant at the Emerson McMillin Observatory.

DATE:— September 7th, 1904.

Sextant Reading	Time Upper Limb	Time Lower Limb	Index Error On the Arc	Index Error Off the Arc
° ′	h. m. s.	h. m. s.	° ′ ″	° ′ ″
66 20	20 03 38.0	20 06 36.0	360 30 00	359 26 50
30	04 05.0	07 05.0	20	50
40	04 33.5	07 33.0	00	45
50	05 02.5	08 00.0		
67 00	05 31.0	08 30.5		

From the Ephemeris for 1904, page 405, we find the declination for Washington mean noon, (δ_0), $= + 6°$ 03′ 36″.1, the hourly change in declination, $(\Delta\delta')$, $= -$ 56″.22, the equation of time for Washington apparent noon, (E_0), $= - 2^m 02^s.63$, and its change in one day, $(\Delta E')$, $= - 20^s.11$. The value of $\Delta E'$ is obtained by subtracting E_0 for the 6th from E_0 for the 7th, since the observations were made in the morning and the date is in civil reckoning. We also have the barometer $= 29″.46$, the attached thermometer $= 68°$ F., the external temperature $= 58°$ F., $\phi = 39°$ 59′ 50″, the approximate value of $\Delta T = + 28^m 36^s$, and $L = 23^m 46^s.8$ west. The reduction is given on the next page.

Reduction: Time, Single Altitude of the Sun

Index Error		Declination Cont'd	
$R_1 =$ . . . . $360°\ 30'\ 07''$		Same in hours . . $20^h.974$	
R_2 . . . . . $359°\ 26'\ 48''$		‡Above — 24^h . — $3^h.026$	
Sum . . . . $719°\ 56'\ 55''$		— $3.027 \times \Delta\delta'$. $+2'\ 50''.2$	
½ Sum . . . $359°\ 58'\ 28''$		δ, ... $+\ 6°\ 03'\ 36''.1$	
*I . . . . . . $+\ 1'\ 32''$		Sum $= \delta$. $+\ 6°\ 06''\ 26''.3$	

Altitude		True Time	
Observed 2h . $66°\ 40'\ 00''$		§Hour angle — $50°\ 50'\ 34''$	
2h + I . . . $66°\ 41'\ 32''$		t in time . — $3^h\ 23^m\ 22^s.3$	
½ (2h + I) . $33°\ 20'\ 46''$		Same + L . . . — $2^h\ 59^m$	
†Refraction . . — $01'\ 25''$		Above in days . . — 0.125	
†Parallax . . . . . + $07''$		— $0.125 \times \Delta E'$ . . + $2^s.5$	
True h . . . $33°\ 19'\ 28''$		E_0 . . . . . . — $2^m\ 02^s.6$	
Z . . . . . . $56°\ 40'\ 32''$		Sum $= E$ . . — $2^m\ 00^s.1$	

Declination		$24^h — t$ . . . $20^h\ 36^m\ 37^s.7$	
Average time $20^h\ 06^m\ 03^s.4$		Mean time . $20^h\ 34^m\ 37^s.6$	
Assumed ΔT . + $28^m\ 36^s.0$		Observed time $20^h\ 06^m\ 03^s.4$	
Longitude . . . $23^m\ 46^s.8$		Diff. $= \Delta T$. + $28^m\ 34^s.2$	
Wash. time . $20^h\ 58^m\ 26^s.2$			

NOTES

*See page 56. †See Chapter III. ‡Twenty-four hours is subtracted from this quantity in order to interpolate from the nearest place given in the Ephemeris. §Computed by formulæ (20) and (21), page 14; the hour angle is negative, since, the observations being made in the morning, the sun is east of the meridian. In reducing his observations the student should prepare a form for this part of the computation.

If the observations are reduced separately, r and E may be computed as above, δ should be computed for the first and last observations and interpolated for the others.

The semi-diameter of the sun should be taken out of the Ephemeris and applied to the true h. If a star is observed, a and δ should be taken from the Ephemeris for the nearest day, E is omitted, and the true sidereal time is given by $T = t + a$.

Time by Double Altitudes of the Sun
Observing Program

The observing program is similar to the preceding, except that half of the observations are made in the morning and the remaining half in the afternoon, the instrument settings being repeated in the reverse order. This method is especially adapted to the sextant, as the delicacy of the alidade level of a theodolite makes it almost impossible to repeat, in the afternoon, the same series of zenith distances observed in the morning. It should be noted that it is not necessary, though advisable, to read the barometer and thermometers or to observe for index error.

Process of Reduction

In the case of a star observed at equal zenith distances east and west of the meridian, we have seen that the clock error is given by $\Delta T = a - \frac{1}{2}(T_e + T_w)$. The equation takes this simple form because the star's declination remains practically constant between the observations, a condition which is not realized in the case of the sun. The equation of time varies so slowly that we may assume it constant between the observations and equal to E_0, its value for local apparent noon, since the observations are very nearly symmetrically distributed about this instant. The true apparent times of the two observations may then be taken as $T_e + \Delta T - E_0$ and $T_w + \Delta T - E_0$, whence we have for the two hour angles, (t_e) and (t_w), $t_e = 24^h - (T_e + \Delta T - E_0)$ and $t_w = T_w + \Delta T - E_0$.

Since the declination varies slowly, for interpolating the value of δ we may assume

(68) $\quad t_e = t_w = t_0 = \frac{1}{2}(t_e + t_w) = 12^h + \frac{1}{2}(T_w - T_e)$

The change in δ between its value for local apparent noon and its value at either observation is then $(\Delta\delta')t_0$, where $(\Delta\delta')$ is the hourly change in declination and t_0 is expressed in hours. This change in the sun's declination causes, evidently, the difference between t_0 and t_e or t_w. Let us call this difference Δt, whence we have

(69) $\quad t_e = t_0 - \Delta t$ and $t_w = t_0 + \Delta t$

It only remains, therefore, to determine Δt in terms of $(\Delta\delta')t_0$. In order to do this consider the triangle PZS, Fig. I, page 6 and apply to it formula (2), page 10; we have then, after simplifying,

(70) $\quad \cos Z = \sin \phi \sin \delta + \cos \phi \cos \delta \cos t$

Here ϕ and Z have the same values at the two observations while δ and t have changed slightly; we may, therefore, compute the effect upon δ of a known change in t by differentiating (70), regarding ϕ and Z as constant. This gives, after solving for Δt,

(71) $\quad \Delta t = (tg\, \phi \cosec t_0 - tg\, \delta_0 \cot t_0)\,[(\Delta\delta')t_0]$

since the declination is to be taken for apparent noon and, in the small term Δt, t_0 may replace t with sufficient accuracy. Substitute this value of Δt in (69), replace t_w and t_0 by their values given above, divide by 15 to convert from arc to time, and we have finally, equation (72),

$$\Delta T = 12^h - \frac{1}{2}(T_e + T_w) + E_0 + \left\{ \frac{tg\, \phi}{\sin t_0} - \frac{tg\, \delta_0}{tg\, t_0} \right\} \frac{\Delta\delta'\,t_0}{15}$$

where $t_0 = 12^h + \frac{1}{2}(T_w - T_e)$ expressed in hours.

As an example of the above method, we may take the set of double altitudes of the sun given on the next page. These were observed with a sextant at the Emerson McMillin Observatory, September. 7th, 1904.

From the Ephemeris for 1904, page 405, we find the declination for Washington apparent noon, (δ), $= + 6°$ 03' 38".0, the hourly variation in declination, ($\Delta\delta'$) $=$ $- 56$".22, the equation of time for Washington apparent noon, (E), $= - 2^m 02^s.63$, and its change in one day, ($\Delta E'$), $= - 20^s.3$; we also have L $= 23^m 46^s.8$ west and $\phi = 39° 59' 50"$. From this we find the declination, (δ_0), and the equation of time, (E₀), for local apparent noon equal, respectively, to $+6° 03' 15".7$ and $- 2^m 02^s.95$.

<div align="center">DATE:— September 7th, 1904.</div>

Sextant Reading	Morning Time Upper Limb	Morning Time Lower Limb	Sextant Reading	Afternoon Time Lower Limb	Afternoon Time Upper Limb
° '	h. m. s.	h. m. s.	° '	h. m. s.	h. m. s.
66 20	20 03 38.0	20 06 36.0	67 00	2 49 46.0	2 52 46.0
30	04 05.0	07 05.0	66 50	50 15.0	53 13.5
40	04 33.5	07 33.0	40	50 44.5	53 43.0
50	05 02.5	08 00.0	30	51 10.5	54 11.0
67 00	05 31.0	08 30.5	20	51 40.0	54 38.5

<div align="center">REDUCTION</div>

T_e . . . . . $20^h 06^m 03^s.4$ ½ Above . . $11^h 29^m 08^s.1$

T_w . . . . . $2^h 52^m 12^s.8$ 12^h — Above $+ 30^m 51^s.9$

$T_w - T_e$ $- 17^h 13^m 50^s.6$ E₀ . . . . . . $- 2^m 03.0$

½ Above $- 8^h 36^m 55^s.3$ Sum . . . . $+ 28^m 48^s.9$

t_0 . . . . . . $3^h 23^m 04^s.7$ t_0 in arc . . . . . $50° 46'$

$T_w + T_e$ . . $22^h 58^m 16^s.2$ t_0 in hours . . . . $3^h.385$

log $t_0{}^h$ . . 0.5295   N₁ . . . . . 1.083 $\log \dfrac{\Delta\delta'}{15}$ $1.1033n$

log $\Delta\delta'$ $1.7499n$ tg δ_0 . . . 9.0254

colog 15   8.8239   tg t_0 . . . 0.0880   log 0.997   9.9987

Sum . . $1.1033n$ Diff. . . . 8.9374   Sum . . 1.1020_n

tg ϕ . . . 9.9238   N₂ . . . . 0.086   Number $- 12^s.7$

sin t_0 . . 9.8891   N₁ — N₂   0.997 ΔT $+ 28^m 36^s.2$

Diff. . . 0.0347

CHAPTER X.

LATITUDE

By far the easiest method of determining latitude is by measuring the altitude of a star as it crosses the meridian. For this purpose a theodolite should be used, as a sextant is not convenient for such work. There should also be a distant mark near the horizon, whose azimuth has been determined by one of the methods of Chapter VIII. This mark should be used for finding the zenith point of the instrument, Z_3 of formulæ (42) and (44), and for placing the instrument in the meridian. The theory of the method is obvious from Fig. 1, page 6, as the figure shows at once that $\phi = \delta \pm Z$, where the plus sign is to be used for stars south of the zenith and the minus sign for those north of it.

Latitude: Star on the Meridian
Observing Program

1. Level the theodolite and test all adjustments.

2. Point on the mark circle right, read both circles and the alidade level. It is only necessary to read one microscope of the horizontal circle.

3. Subtract the azimuth of the mark from the horizontal circle reading and set the horizontal circle at the resulting angle, which is called the meridian setting. The horizontal axis will then be east and west and the vertical wire of the eye-piece will mark the meridian.

4. Observe half the stars in this position of the instrument, bisecting them with the horizontal wire as they cross the vertical wire, and reading the vertical circle and its level. Do not try to make the bisections exactly as the star is on the vertical wire, but ten or fifteen seconds before it comes to it; then watch the star as it travels along the wire, trying for an average position, as on most nights the star will not travel smoothly but will appear to wobble back and forth across the wire.

5. Reverse the instrument 180° in azimuth and observe the rest of the stars. The stars should be so placed that half will be observed circle right and half circle left; the above program may not accomplish this and it may be necessary to reverse more than once.

6. Set on the mark circle left, and read vertical circle.

7. Read barometer and thermometers several times during the night's work.

In place of (2) and (6) it is better to observe on the mark, in both positions of the instrument, several times during the night's work, coming back to the meridian setting after each; each of these double pointings gives a value of the zenith point, the agreement of which serves as a check on the work, the mean being used in the final reductions. An observing list should be prepared, which should give the star's name, its magnitude, its right ascension, and the approximate vertical circle reading as it crosses the meridian. This latter may be computed by the following formulæ:

$$Cl = Z_s \mp (\phi - \delta) \text{ and } Cr = Z_s \pm (\phi - \delta)$$

The upper sign should be used for south stars and the lower one for north stars. Z_s should be determined to the nearest minute, neglecting level readings, and for this purpose may be considered constant. When observing,

the vertical circle is set at this reading a few minutes be-
fore the sidereal time equal to the star's right ascension,
and the observer watches for the star as it glides into the
field of view.

Process of Reduction

1. Compute the star's zenith distance by the formulæ
on page 70.

2. Correct for refraction as in Chapter III.

3. Pick out δ from the Ephemeris and compute ϕ by
the formula $\phi = \delta \pm (Z + r)$.

The following meridian altitudes were observed with
the twelve inch theodolite of the Emerson McMillin Ob-
servatory on July 24th, 1904.

Observing List

No.	Star's Name	N S	Mag.	Right Ascension h. m.	Setting Circle Right ° ′	Setting Circle Left ° ′
1	Beta Lyræ	S	3.6	18 47	96 43	83 15
2	Beta Cygni	S	3.1	19 27	102 13	77 45
3	Gamma Sagittæ	S	3.6	19 55	110 45	69 13
4	Theta Aquilæ	S	3.3	20 06	131 05	48 53

Record of Observations

Mark or Star	C	Circle Reading ° ′	Mic. C ′ ″	Mic. D ′ ″	Level R. L. d d
Mark	L	355 45	1 35.1	1 43.1	28.1 04.9
1	L	83 10	4 40.7	4 51.6	30.0 07.3
2	R	102 10	3 09.1	3 16.8	33.5 09.5
3	R	110 40	4 36.3	4 45.7	35.0 10.9
4	L	48 50	3 51.1	4 04.2	26.0 01.7
Mark	R	184 10	1 34.8	1 47.3	30.8 07.3

We also have the following constants: the barometer
read 29″.42, the attached thermometer read 73° F., and

the external temperature was 54° F.; the level was grad-
uated from the left end and D = 2".o. The error of runs
is the microscope reading in minutes times + o".50 for C
and + o".07 for D. From the above data we find Z_8 =
89° 58' 35".o; the rest of the computation, except that
of refraction, is given below.

Reduction

Star	C	Zenith Distance	Ref.	Declination	Latitude
		° ' "	"	° ' "	° ' "
1	L	6 44 24.3	6.8	+ 33 15 21.7	39 59 53.3
2	R	12 13 56.4	12.3	+ 27 45 45.6	· 53.8
3	R	20 50 22.2	21.5	+ 19 14 09.5	52.4
4	L	41 05 03.3	49.3	— 01 06 05.5	47.7

Latitude: Circum-meridian Altitudes

Theory

In the preceding method we obtained the latitude from
the measured altitude of a star through the formula ϕ =
$\delta \pm (90° - h + r)$, h being the observed altitude and r
the refraction. If we had measured the altitude a short
time before or after meridian passage, the above equation
would no longer hold true but we would have to add to
h a small correction x. Obviously x must depend upon
the star's distance from the meridian, e. i., its hour angle,
which is easily obtained, provided we know the error of
our chronometer and note the instant at which we meas-
ured the altitude; t, therefore, is known and it only be-
comes necessary to show how to compute the value of x
in terms of t. For this purpose consider the triangle PZS,
Fig. 1, page 6, where we have PZ = 90° — ϕ, PS = 90°
— δ, and the angle ZPS = t. Apply formula (2), page
10, to it, simplify, and we have

$$\cos Z = \sin \phi \sin \delta + \cos \phi \cos \delta \cos t$$

Substitute $1 - 2 \sin^2 \tfrac{1}{2}t$ for cos t, reduce, and we obtain

$$\cos Z = \cos(\phi - \delta) - 2 \cos \phi \cos \delta \sin^2 \tfrac{1}{2}t$$

Put $y = 2 \sin^2 \tfrac{1}{2}t$ and we have

(73) $\cos Z = \cos(\phi - \delta) - y \cos \phi \cos \delta$

This equation tells us that Z is a function of y, ϕ and δ being constant; moreover, since the observations should always be made within 30^m of the time of meridian passage, y is a small quantity. Let us rewrite equation (73) as follows:

(74) $\cos \psi(y) = \cos(\phi - \delta) - y \cos \phi \cos \delta$

Apply to (74) Maclaurin's Theorem, which states that if y is a small quantity we may expand $\psi(y)$ as follows:

(75) $\psi(y) = \psi(o) + \psi'(o) y + \tfrac{1}{2}\psi''(o) y^2$

where $\psi(o)$, $\psi'(o)$, and $\psi''(o)$ denote the value of $\psi(y)$ and its several derivatives when y has the value zero. When $y = o$, equation (74) becomes

$$\psi(o) = \phi - \delta$$

Differentiate (74) and we have

(76) $- \sin \psi(y) \, \psi'(y) = - \cos \phi \cos \delta$

Put $y = o$ and solve for $\psi'(o)$, and we obtain

$$\psi'(o) = \frac{\cos \phi \cos \delta}{\sin \psi(o)} = \frac{\cos \phi \cos \delta}{\sin(\phi - \delta)}$$

Differentiate (76) and we have

$$- \sin \psi(y) \, \psi''(y) - \cos \psi(y) \, [\psi'(y)]^2 = o$$

Put $y = o$, substitute the value of $\psi'(o)$, solve for $\psi''(o)$, and we obtain, after simplifying,

$$\psi''(o) = - \cot (\phi - \delta) \frac{\cos^2 \phi \cos^2 \delta}{\sin^2(\phi - \delta)}$$

Substitute the values of $\psi(o)$, $\psi'(o)$, and $\psi''(o)$ in (75); for $\psi(y)$ put its value Z, replace y by its value $2 \sin^2 \tfrac{1}{2}t$, divide the last two terms by the sin $1''$ to convert from radians to arc, and we have

$$(77) \quad \phi = Z + \delta - \frac{\cos \phi \cos \delta}{\sin(\phi - \delta)} \times \frac{2 \sin^2 \frac{1}{2}t}{\sin 1''} +$$

$$\cot(\phi - \delta) \frac{\cos^2 \phi \cos^2 \delta}{\sin^2(\phi - \delta)} \times \frac{2 \sin^4 \frac{1}{2}t}{\sin 1''}$$

In equation (77) let us make the following substitutions: $\frac{\cos \phi \cos \delta}{\sin(\phi - \delta)} = A$, $\frac{2 \sin^2 \frac{1}{2}t}{\sin 1''} = m$, $\frac{2 \sin^4 \frac{1}{2}t}{\sin 1''} = n$, and $A^2 \cot(\phi - \delta) = B$; with these changes (77) becomes

$$(78) \quad . \phi = Z + \delta - Am + Bn$$

The above equation was derived upon the supposition that the star is south of the zenith; if it is north of the zenith and above the pole, (78) should be replaced by

$$(79) \quad \phi = \delta - Z + Am - Bn$$

and, if below the pole, by

$$(80) \quad \phi = 180° - \delta - Z - Am - Bn$$

The values of m and n are given in Tables V and VI for the argument t. A and B both contain the unknown latitude, ϕ, but, as these terms are both small, it is only necessary to have an approximate value of ϕ to compute their values. Now suppose N observations were made in rapid succession; each observation would give rise to an equation of the form $\phi = Z_e + \delta_e - A_e m_e + B_e n_e$, where the subscript e takes the values 1, 2, 3, . . . N. Let us average this system of equations and we will have

$$(81) \quad \phi = \frac{\Sigma Z_e}{N} + \frac{\Sigma \delta_e}{N} - \frac{\Sigma A_e m_e}{N} + \frac{\Sigma B_e n_e}{N}$$

In the above equation the first term is simply the average observed zenith distance, corrected for refraction and parallax, and may be represented by Z_1; the second term is the average declination and may be represented by δ_1. If a star is observed, δ_1 may be picked out for the nearest day; if the sun is observed, δ_1 may be picked out for the average time since it varies proportionally with the time.

A and B, represented by A_1 and B_1, may be considered constant and computed with δ_1. Substituting these values in (81), we have finally

$$(82) \qquad \phi = Z_1 + \delta_1 - A_1\frac{\Sigma \, m_e}{N} + B_1\frac{\Sigma \, n_e}{N}$$

This equation applies to stars south of the zenith only and should be modified as above for stars at other points.

To find the Value of the Hour Angle, t.

1. Sun observed with a mean time chronometer.

(a) As explained on pages 30 and 31, compute the equation of time, (E_1), for the average of the times. The sun's hour angle is simply the apparent time and is given by the equation $t = T + \Delta T - E$, in which E may be replaced by E_1 without sensible error. Compute $E_1 - \Delta T$, or better $24^h + E_1 - \Delta T$, once for all, and then the hour angle for each observation by $t = T - (24^h + E_1 - \Delta T)$. If any T is past chronometer noon, add 24^h.

(b) Compute the value of E for local apparent noon. $T + \Delta T - E$ will then no longer be the sun's hour angle, because the chronometer does not keep apparent time. Let $t' = T - (E - \Delta T)$; since t' differs but little from t, we may assume that $\sin \frac{1}{2}t : \sin \frac{1}{2}t' :: t : t'$, whence $\sin^2 \frac{1}{2}t = (t \div t')^2 \sin^2 \frac{1}{2}t'$. But, $t : t' :: 24^h : 24^h + \Delta' E$, where $\Delta' E$ is the change in E in 24^h. Let us put $k = [24^h \div (24^h + \Delta' E)]^2 = [86400^s \div (86400^s + \Delta' E^s)]^2$, and we will have $\sin^2 \frac{1}{2}t = k \sin^2 \frac{1}{2}t'$. If, therefore, we replace A_1 by kA_1, we may use t' in picking out m and n. The logarithm of k is given in Table VII.

2. Sun observed with a sidereal chronometer.

(a) Pick out a for the average of the times and compute t by the formula $t = T - (a - \Delta T)$. This method should not be used if the time from the first to the last observation is more than ten minutes.

(b) Pick out a for local apparent noon and compute $t' = T — (a — \Delta T)$; here t' is expressed in sidereal h. m. s., and exactly as above it may be shown that we may pick out m and n with t' as argument, provided we replace A_1 in formula (82) by $0.99455kA_1$. [See page 27.]

3. Star observed with a sidereal chronometer. Compute t by the formula $t = T — (a — \Delta T)$.

4. Star observed with a mean time chronometer. Compute T_0, the mean time the star is on the meridian, by the rules on page 30. Compute t' by the formula $t' = T — (T_0 — \Delta T)$; this gives t' in mean time h. m. s. whereas it should be in sidereal h. m. s. Pick out m and n with t' as argument and replace A_1 by $1.00548kA_1$.

In all of the above cases, if the chronometer has a daily rate greater than about 2^s for refined work and 10^s for rough work, the argument used in finding k should be $\Delta'E — r$ for the sun, and r for the stars. Where k does not appear it should be introduced. In the above equations the quantity $Am — Bn$ is called the reduction to the meridian.

Observing Program: Sextant

1. Test all adjustments and observe for index error.

2. Make three to five pointings on one limb of the sun, recording the corresponding times and sextant readings.

3. Repeat (2) on the opposite limb of the sun.

4. Read barometer and thermometers.

Observing Program: Theodolite

The observing program for a theodolite is the same as the above, except that, in place of the index error, Z_3 should be determined if the observations are to be reduced separately. The alidade level should be read after each observation.

Process of Reduction

1. Average the times and, for a sextant, the observed

2h; for a theodolite, compute Z by formula (43) or (74).

2. Correct for index error, refraction, and parallax.

3. Find δ_1 as in (2) and (3), page 85.

4. Compute the hour angle by one of the methods given on pages 107 and 108, and ϕ by formula (82).

If the observations are reduced separately, it will be necessary to pick out the sun's semi-diameter from the Ephemeris, and add it to or subtract it from the measured altitude according to the limb on which the pointings were made. A different value of the declination should be computed for each observation, and a separate value of the refraction for the two limbs of the sun. A_1 and B_1 may be considered constant and computed with the average value of the declination.

The following observations of the sun for latitude were made at the Emerson McMillin Observatory with sextant number 5971 and mean time chronometer P. & F.

DATE:— March 23rd, 1904.

Observed Double Altitude	Limb	Observed Time	Hour Angle	m	Index Error
° ′ ″		h. m. s.	m. s.	″	° ′ ″
102 39 20	U	23 38 22	0 42	1.0	360 30 10
38 30	U	40 10	2 30	12.3	00
38 00	U	41 21	3 41	26.6	10
101 32 40	L	43 18	5 38	62.3	359 26 35
31 05	L	44 44	7 04	98.0	00
29 30	L	46 25	8 45	150.3	10

From the Ephemeris for 1904, page 401, we find δ_0 $= + 1° \, 03' \, 17''.5$, the hourly change in the declination, $(\Delta'\delta)$, $= + 59''.11$, the change in E in one day, $(\Delta'E)$, $= - 18^s.28$, and $E_0 = + 6^m \, 42^s.6$. We also have given $\Delta T = + 29^m \, 02^s$, $L = 23^m \, 46^s.8$ west, and the approx-

imate $\phi = 39° 59' 50''$; the barometer reads $29''.55$, the attached thermometer reads $66°.2$ F., and the external temperature is $50°.5$ F.

Reduction

Average 2h	102° 04' 51"	ΔE = 0.02 Δ'E	— 0ˢ.4
*I	+ 1' 49"	E₀	+ 6ᵐ 42ˢ.6
Sum	102° 06' 40"	Sum = E₁ . .	+ 6ᵐ 42ˢ.2
Observed h . .	51° 03' 20"	24ʰ + E₁ . .	24ʰ 06ᵐ 42ˢ.2
†Refraction	— 46"	ΔT	+ 29ᵐ 02ˢ.0
†Parallax	+ 06"	‡Difference	23ʰ 37ᵐ 40ˢ
True h	51° 02 40"	φ — δ	38° 55' 58"
Average time	23ʰ 42ᵐ 23ˢ	cos φ	9.88427
ΔT + L . . .	+ 52ᵐ 49ˢ	cos δ	9.99992
Wash. M. T. .	0ʰ 35ᵐ 12ˢ	Sum	9.88419
Same in hours . . .	0ʰ.59	sin(φ — δ) . . .	9.79825
Δδ = 0ʰ.59Δ'δ . .	+ 34".9	Difference	0.08594
δ₀	+ 1° 03' 17".5	log k	9.99982
Sum = δ	+ 1° 03' 52"	log Σm ÷ N . . .	1.76641
Wash. M. T. . .	0ʰ 35ᵐ 12ˢ	Sum	1.85217
E₀	+ 06ᵐ 43ˢ	kA₁ ×(Σm ÷ N)	1' 11"
Wash. A. T. . .	0ʰ 28ᵐ 29ˢ	δ	+ 1° 03' 52"
Same in days	0.02	True Z . . .	38° 57' 20"
		φ	40° 00' 01"

NOTES

*See (6), page 55. † See Chapter III. ‡ This quantity subtracted from each observed time gives the corresponding hour angle; n is negligible in this example.

Latitude from Polaris
Theory

For a theodolite, one of the best methods of finding latitude is to measure the altitude of Polaris at a known instant of time. Let T be the observed chronometer time,

ΔT the error of the chronometer on sidereal time, t the hour angle, a the right ascension, and h the observed altitude of Polaris. We will then have

(83) $\qquad t = T + \Delta T - a$

In the triangle PZS, $PZ = 90^\circ - \phi$, $PS = 90^\circ - \delta$, $ZS = 90^\circ - h$, and the angle $ZPS = t$. Apply formula (2) to this triangle, put c for $90^\circ - \delta$, reduce, and we get

(84) $\qquad \sin h = \sin \phi \cos c + \cos \phi \sin c \cos t$

But $\cos t = 1 - 2 \sin^2 \tfrac{1}{2} t$; substitute this in (84), reduce, and we have

(85) $\quad \sin(\phi + c) = \sin h + 2 \cos \phi \sin c \sin^2 \tfrac{1}{2} t$

Equation (85) tells us that ϕ is a function of c, h and t being constant. We have, therefore, $\phi = \psi(c)$. In the case of Polaris c is always small and we may write

(86) $\quad \phi = \psi(c) = \psi(o) + \psi'(o) c + \tfrac{1}{2} \psi''(o) c^2$

where $\psi(o)$, $\psi'(o)$, and $\psi''(o)$ denote the values of $\psi(c)$ and its several derivatives when c has the value zero.

We may rewrite (85) as follows:—

(87) $\quad \sin[\psi(c) + c] = \sin h + 2 \cos \psi(c) \sin c \sin^2 \tfrac{1}{2} t$

When $c = o$, (87) becomes $\sin \psi(o) = \sin h$, or $\psi(o) = h$. Differentiate (87) and we obtain

(88) $\quad [\psi'(c) + 1] \cos [\psi(c) + c] =$
$$- 2 \psi'(c) \sin \psi(c) \sin c \sin^2 \tfrac{1}{2} t +$$
$$2 \cos \psi(c) \cos c \sin^2 \tfrac{1}{2} t$$

When $c = o$, equation (88) becomes

$[\psi'(o) + 1] \cos \psi(o) = 2 \cos \psi(o) \sin^2 \tfrac{1}{2} t$, whence
$$\psi'(o) = - 1 + 2 \sin^2 \tfrac{1}{2} t = - \cos t$$

Differentiate equation (88) and we have

$- [\psi'(c) + 1]^2 \sin[\psi(c) + c] + \psi''(c) \cos [\psi(c) + c] =$
$$- 2 \sin^2 \tfrac{1}{2} t [(\psi'(c))^2 \cos \psi(c) \sin c +$$
$$2 \psi'(c) \sin \psi(c) \cos c + \psi''(c) \sin \psi(c) \sin c +$$
$$\cos \psi(c) \sin c]$$

When $c = o$, $\psi(c) = h$, $\psi'(c) = - \cos t$, and we have

$-(1 - \cos t)^2 \sin h + \psi''(0) \cos h = 4 \sin h \cos t \sin^2 \frac{1}{2} t$
Solve for $\psi''(0)$, simplify and reduce, and we have $\psi''(0)$
$= tg\ h \sin^2 t$. Substitute the values of $\psi(0)$, $\psi'(0)$, and
$\psi''(0)$ in (86) and we obtain
$\phi = \psi(c) = h - c \cos t + \frac{1}{2} c^2\ tg\ h \sin^2 t$, or in arc
(89) $\phi = h - c \cos t + [4.3845]\ c^2\ tg\ h \sin^2 t$
where $4.3845 = \log \frac{1}{2} \sin 1''$. The terms neglected in
(89) never amount to more than $1''$ within the limits of
the United States.

Observing Program

The observing program for a theodolite is exactly the
same as that given under azimuth from Polaris, except
that the vertical circle is read in place of the horizontal
circle and the alidade level in place of the striding level.
This level should be read after each observation and the
barometer and thermometers at least once during the
night's work. If the observations are to be reduced by
the method of averages the readings on the mark may be
omitted. If a sextant is used the index error should be
determined.

Process of Reduction

1. Compute Z by the formulæ on page 70.
2. Correct for refraction.
3. Pick out of the Ephemeris a and δ for the nearest
tenth of a day.
4. Compute t by formula (83) and ϕ by formula (89).

As an example of the above method we may take the
observations of Polaris given on the next page.

From the Ephemeris for 1904, page 321, we find $a =$
$1^h\ 25^m\ 57^s.5$ and $\delta = + 88° 47' 47''.9$; we have also the
barometer $= 29''.42$, the attached thermometer $= 65°$ F.,
and the external temperature $= 44°$ F. With these data
we find $c = 4332''.1$, $\log c = 3.6367$, $\log c^2 = 7.2734$, and

log [4.3845] c^2 = 1.6579. We also have ΔT = + 32^s.5; the error of runs is the microscope reading in minutes multiplied by + 0″.50 for C and + 0″.07 for D; the level reads from the left and D = 2″.0.

DATE:— October 3rd, 1904.

Object	C	Time	Circle Reading	Mic. C	Mic. D	Level R L
		h. m. s.	° ′	′ ″	′ ″	d. d.
Mark	R		184 10	1 53.2	2 23.8	31.0 4.8
Polaris	R	20 08 16	139 45	0 20.0	0 42.0	34.0 7.2
Polaris	L	20 11 21	40 10	3 42.1	4 09.0	31.0 4.1
Mark	L		355 45	1 11.0	1 39.4	34.5 7.0

Reduction

	Circle Right	Circle Left
Z. See page 70.	49° 46′ 11″.2	49° 45′ 17″.2
h = 90° Z . . .	40° 13′ 48″.8	40° 14′ 42″.8
Refraction . . .	1′ 08″.3	1′ 08″.3
h — r	40° 12′ 40″.5	40° 13′ 34″.5
Observed T . . .	20^h 08^m 16^s.0	20^h 11^m 21^s.0
T + ΔT — a = t	18^h 42^m 50^s.7	18^h 45^m 55^s.7
t in arc	282° 42′ 45″	281° 29′ 00″
sin t	9.9924n	9.9912n
sin^2 t	9.9848	9.9824
tg h	9.9274	9.9276
log 3rd term . .	1.5701	1.5679
3rd term	37″.2	37″.0
cos t	9.2693	9.2990
log(c cos t) . .	2.9060	2.9357
c cos t	+ 805″.4	+ 862″.4
ϕ	39″ 59′ 52″.4	39° 59′ 49″.1

The method of averages gives ϕ = 39° 59′ 50″.7.

CHAPTER XI.

The Astronomical Transit

The astronomical transit is in a general way similar to the theodolite except that it has no horizontal circle and only a rough vertical circle. The eye-piece is provided with a system of five or more vertical wires and two horizontal wires. This system of spider lines is called the reticle. The telescope is much larger than on most theodolites, being from two and one-half to three inches in aperture in field instruments. The instrument is mounted so that the horizontal axis shall point as nearly as possible due east and west and be level. The line of sight will then very nearly mark out the meridian, and, if we note on a sidereal chronometer the time, T, at which a star crosses the middle wire, we.will have $T + \Delta T = a$, and ΔT at once becomes known.

So far as they apply, all of the adjustments of the theodolite should be repeated for the transit; there is one other, however, which is of great importance, namely, the adjustment whereby the axis is placed east and west. This is called "placing the instrument in the meridian" and is easily done as follows: first, place it as nearly as possible in the meridian by eye and level it carefully; then observe the time of transit of a star near the zenith, or better, of two stars, one north and one south of the zenith, both near to it, and of approximately equal zenith dis-

tances. From this we can obtain a close approximation to the true clock error by the formula $\Delta T = a -- T$. With this value of ΔT compute the time by the chronometer at which a close circum-polar star will be on the meridian. A few minutes before this computed time move the instrument in azimuth until the star is bisected by the middle cross-wire and keep following it by moving the azimuth adjusting screws until the computed time of its meridian passage. Repeat this several times until zenith stars and circum-polar stars give nearly the same clock error.

Theory of the Transit

     With all due care there will be three errors in the adjustment of the transit; namely, the want of level of the horizontal axis (b), the error of collimation (c), and the angle the horizontal axis makes with the east and west line (a). It therefore becomes necessary to compute the effect of each of these errors upon the observed time of a star's transit. In Fig. XV, let $H_1 H_2 H_3$ be the
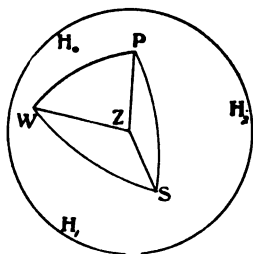
horizon, Z the zenith, P the pole, S the star, and W the point where the prolongation of the west end of the horizontal axis cuts the celestial sphere. In the triangle ZPS, the angle $ZPS = t$, the star's hour angle when it is on the middle wire or the error in the observed time due to the combined effect

FIG. XV.   of a, b, and c. The arc WP would equal $90°$ if a and b were both zero, but, since they are not, WP will equal $90° - n$, where n is a small quantity. In the same way we will put $90° - m$ for the angle WPZ. Obviously, we have $WZP = 90° + a^1$ and $WS = 90° + c$.

1 It is customary to count "a" positive when "W" is south of west.

Collecting these results, we will have

$$WP = 90° — n \qquad PS = 90° — \delta$$
$$WPZ = 90° — m \qquad ZP = 90° — \phi$$
$$WZP = 90° + a \qquad ZPS = t$$
$$WS = 90° + c \qquad WPS = 90° — m + t$$
$$WZ = 90° — b \qquad \text{Here t is reckoned positive to the east.}$$

Apply formula (2), page 10, to the triangle WPS, reduce, and we obtain ·

(90) $\quad$ — sin c = sin n sin δ + cos n cos δ sin(m — t)

In the above, c, n, m, and t are all small quantities and we may replace their sines by their arcs and their cosines by unity; equation (90) then becomes

$$— c = n \sin \delta + (m — t)\cos \delta$$

Solving the above for t, we find

(91) $\qquad$ t = m + n tg δ + c sec δ

It is to be noted that n is the declination and 90° — m is the hour angle of the west end of the axis, while a and b are respectively the altitude and azimuth of that same point. Now it happens that a and b are easier to determine than m and n; and, therefore, if we apply to these quantities the principles of the transformation of co-ordinates, we may determine m and n in terms of a and b. Apply formulæ (2) and (5), pages 10 and 11, to the triangle WPZ, simplify, reduce, and we obtain

$$n = b \sin \phi — a \cos \phi$$
$$m = b \cos \phi + a \sin \phi$$

Substitute these values of m and n in (91), reduce, and we will have

(92) $\quad t = a \dfrac{\sin(\phi — \delta)}{\cos \delta} + b \dfrac{\cos(\phi — \delta)}{\cos \delta} + \dfrac{c}{\cos \delta}$

This equation may be written as follows:

(93) $\qquad$ t = A a + B b + C c

In equation (93), A, B, and C are the same on all nights

but different for each star, while a, b, and c are the same for all stars but vary from night to night. We also have, since here t is positive to the east, $t = a — (T + \Delta T)$, and this combined with (93) gives at once

(94) $\Delta T = a — (T + A a + B b + C c)$

One other correction must still be taken into account, namely, the diurnal aberration. It has been shown in Chapter V that a star's position is shifted by an amount equal to the velocity of the observer divided by the velocity of light. The earth in revolving on its axis carries the observer with it, and it can be easily shown that his velocity is equal to $v_e \cos \phi$, v_e being the velocity of the earth's equator. This increases a star's right ascension by an amount given by the equation $\Delta a = 0^s.021 \cos \phi \sec \delta$.[1] Introducing this correction into (94), we have finally

(95) $\Delta T = a + \Delta a — (T + A a + B b + C c)$

A number of stars can be seen twice in each twenty-four hours as they cross the meridian, once above pole at upper culmination, [U. C.], and once below pole at lower culmination, [L. C.]. For lower culmination stars we should replace δ by $180° — \delta$ in formula (95).

Determination of the Constants a, b, and c.

In the derivation of formula (95) we have assumed that b is to be plus when the west end of the axis is too high. Formulæ (56) and (57), page 77, apply directly if we replace the word "circle" by the word "west" and the word "opposite" by the word "east". It remains, however, to consider the effect of a difference in size of the two pivots, F_1 and F_2, Fig. X, page 60. We shall sonsider only that case in which the Ys of the level and those of the uprights, D_1 and D_2, have the same angle, as is generally the case. Let Figs. XVI and XVII repre-

1 See Doolittle's "Practical Astronomy" pages 289. 290, and 291.

sent two positions of the horizontal axis: the first, circle west; the second, circle east.

The following notation applies to both figures:—

FIG. XVI.

P_1 and P_2, the two pivots.

AL and AZ, the lines joining the tops of the pivots.

LR, the line joining the tops of the standards, D_1 and D_2, Fig. X, page 60.

ML and MZ, the lines bisecting the angles ALR and AZL.

LK, the true level line.

FIG. XVII.

In this discussion we shall consider that the Ys in both level and standard are replaced by flat surfaces. This is allowable, since the Y angles being the same nothing would be altered, if at the same time we imagined flat Ys we would replace each pivot by an imaginary pivot whose

diameter is equal to the distance between two lines, one
of which joins the points of contact of the actual pivot and
the level Y, and the other the points of contact of the act-
ual pivot and the standard Y.

In the first position of the instrument our level read-
ings give us the inclination of the line, LA, joining the
tops of the pivots, i. e., the angle ALK; call this B_w. In
the derivation of formula (95), b was defined as the incli-
nation of the line joining the centers of the pivots, i. e.,
MLK; call this b_w. Let $K = RLK$ and $p = MLA = MLR$. We will then have

$$b_w = MLK = MLR + RLK = K + p$$
$$(96) \quad B_w = ALK = ALR + RLK = K + 2p, \text{ whence}$$
$$(97) \quad b_w = B_w - p$$

Now suppose the instrument to be reversed in its Ys
as is shown in Fig. XVII. Draw ZI parallel to LK, pro-
long AZ to T, MZ to S, and LZ to R. If we are careful
in reversing we may assume that the inclination of the
line joining the tops of the uprights remains unchanged,
i. e., $K = ZLK = RZI$. As above, our level readings
give us $B_e = TZI$ and we require $b_e = SZI$. We also
have $p = SZR = SZT$, whence

$$b_e = SZI = RZI - SZR = K - p$$
$$(98) \quad B_e = TZI = RZI - RZT = K - 2p, \text{ whence}$$
$$(99) \quad b_e = B_e + p$$

From (96) and (98) we find

$$(100) \quad p = \tfrac{1}{4} (B_w - B_e)$$

If the pivots are not round p will vary with each set-
ting of the instrument, and the astronomer should make
a number of determinations of p for varying zenith dis-
tances, plot the results on cross section paper, and draw
a smooth curve through them. From this curve he can
take off the value of p for any setting of the instrument

corresponding to any given determination of B_e or B_w, and compute the values of b_e and b_w by formula (97) or (99).

Example

Setting:— Objective 30° north, circle east. Observer, C.

Level Readings			
Circle West		Circle East	
West End	East End	West End	East End
19.9	49.8	19.4	49.5
48.3	18.2	48.7	18.7

The level is graduated from one end. Apply formula (56), page 77, as modified above, and we find $B_w = -0.80$, $B_e = -0.38$, whence $p = \frac{1}{4}[-0.80 - (-0.38)] = -0.105$. Applying formulæ (97) and (99), we find $b_w = -0.695$ and $b_e = -0.485$.

In formula (95) all terms are now known except the first, fourth, and sixth, and we may write

(101) $\Delta T + A a + C c = M$

where M is a known quantity. Each star gives an equation of this form; and if, for example, we were to observe twelve stars, we would have twelve equations containing the three unknown quantities ΔT, a, and c, and we could determine their most probable values by the method of Least Squares. For most work such a refinement is unnecessary and we may use the following easier method. Since A depends only upon ϕ and δ, we may select two stars, one north and the other south of the zenith, which satisfy the condition $A_1 = -A_2$, or

$$\frac{\sin(\phi - \delta_1)}{\cos \delta_1} = -\frac{\sin(\phi - \delta_2)}{\cos \delta_2}$$

From this we have

$$\mathrm{tg}\ \delta_1 + \mathrm{tg}\ \delta_2 = 2\ \mathrm{tg}\ \phi$$

If the two stars nearly satisfy this condition, $\frac{1}{2}(A_1 + A_2)$ will nearly equal zero, and in the average of the two equations may be dropped as being insignificant. Whence

Emerson McMillin Observatory, July 19th, 1904. Time with the Transit.

Star's name	Beta Draconis West	Alpha Ophiuchi West	Iota Herculis East	Mu Herculis East	35 Draconis East
Circle					
Level readings $\{$	w 12.0 E 37.0 / 43.5 18.3			w 15.3 E 40.9 / 38.8 13.0	w 38.5 E 13.0 / 16.5 42.0
$\Sigma L_e - \Sigma L_w$	+ 12.8			− 04.4	− 07.0
b in level divisions	+ 3.20			− 1.10	− 1.75
p " "	+ 0.23			+ 0.22	+ 0.38
$b^s = 0^s.1(b \pm p)$	+ 0^s.30			− 0^s.09	+ 0.38
A	− 0.35	+ 0.47	− 0.15	+ 0.24	− 2.67
B	+ 1.60	+ 0.91	+ 1.43	+ 1.10	+ 3.55
C	+ 1.64	+ 1.02	− 1.44	− 1.13	− 4.44
Average time	h. m. s. 17 27 56.06	h. m. s. 17 30 08.60	h. m. s. 17 36 24.60	h. m. s. 17 42 21.48	h. m. s. 17 53 25.18
B b	+ 00.48	+ 00.27	− 00.16	− 00.12	− 00.39
T + B b	17 27 56.54	17 30 08.87	17 36 24.44	17 42 21.36	17 53 24.79
$\alpha + \Delta a$	17 28 18.26	17 30 31.22	17 36 47.80	17 42 44.59	17 53 48.71
$\alpha + \Delta a - (T + B b)$	+ 21.72	+ 22.35	+ 23.36	+ 23.23	+ 23.92
A a	− 00.12	+ 00.16	− 00.05	+ 00.08	− 00.89
C c	− 00.79	− 00.49	+ 00.69	+ 00.54	+ 02.13
ΔT	+ 22.63	+ 22.68	+ 22.72	+ 22.61	+ 22.68

Alpha Ophiuchi: Level was not read on this star. Use preceding value of b.

Iota Herculis: Level was not read on this star. Use the mean of the next two values for all stars, circle east.

we will have for the average of the two

$$\Delta T + \tfrac{1}{2}(C_1 + C_2)c = \tfrac{1}{2}(M_1 + M_2)$$

Now reverse the instrument and observe a similar pair of stars. By reversing we have changed the sign of c and we will have from the second pair

$$\Delta T - \tfrac{1}{2}(C_1' + C_2')c = \tfrac{1}{2}(M_1' + M_2')$$

From the above equations we may find an approximate value of c and ΔT. Now observe a star near the pole, preferably at L. C., and we will have

$$\Delta T + A_3 a \pm C_3 c = M_3$$

Here A_3 is large and hence any error in the approximate values of c and ΔT will have a comparatively small effect upon the value of a derived from this equation by using the values of c and ΔT already found. We will then have

$$a = [M_3 - (\Delta T \pm C_3 c)] \div A_3$$

The sign before $C_3 c$ is determined by the position of the circle; thus c may always be taken as plus when the circle is west and minus when it is east. As a check, the values of a and c should be substituted and ΔT computed for each star. If north and south stars give the same ΔT, a is correct; and if circle east and circle west stars give the same ΔT, c is correct; if not, make a second approximation, using the value of a already found in finding c and ΔT, and with their new values recompute a.

Example

On the preceding page is given a complete example. The values of A, B, C, p, and Δa were taken from special tables prepared for the observatory, and the right ascensions from the " Berliner Astronomisches Jahrbuch " for 1904. The computation of a and c is as follows: the first two stars give the equations

$$\Delta T - 0.35 a + 1.64 c = + 21^s.72$$
$$\Delta T + 0.47 a + 1.02 c = + 22^s.35$$

Averaging these we find

(1) $\Delta T + 0.06\,a + 1.33\,c = + 22^s.04$

The third and fourth stars give

$\Delta T - 0.15\,a - 1.44\,c = + 23^s.36$

$\Delta T + 0.24\,a - 1.13\,c = + 23^s.23$

Averaging the above we will have

(2) $\Delta T + 0.04\,a - 1.28\,c = + 23^s.30$

Since a is kept under 1^s, the second term in (1) and (2) may be neglected and we will have

$\Delta T + 1.33\,c = + 22^s.04$

$\Delta T - 1.28\,c = + 23^s.30$

The solution of the above gives $c = -0^s.48$ and $\Delta T = + 22^s.68$. With this value of c we find, for 35 Draconis, that $C\,c = + 2^s.13$, whence we obtain from this star

$+ 22^s.68 - 2.67\,a + 2^s.13 = + 23^s.92$

This gives $a = + 0^s.33$. These values of a and c were used in the final reduction given on page 122; and it will be noticed that the check for a and c, given above, is satisfied to $0^s.02$, showing that a second approximation is entirely unnecessary.

The times given in the preceding example are the means of the times of transit over five vertical threads, each transit being observed to tenths of a second. It sometimes happens that an observer will miss his observation on one or more threads but secure them on all of the rest. In this case it becomes necessary to correct his average time of transit for the missing thread, which can be easily done by reducing all the observations to the middle thread by the following method. Let $i_1, i_2, i_3, \ldots i_n$ be the angles between each thread and the middle thread as seen from the center of the objective. These angles are exactly similar to the collimation, (c), and hence the times, $I_1, I_2, I_3, \ldots I_n$, that a star takes in passing from

any thread to the middle thread will be given by

(102) $I_1 = i_1 \sec \delta$, $I_2 = i_2 \sec \delta$, ... $I_n = i_n \sec \delta$

If we average the above we will have the time it takes a star to pass from the mean or average thread to the middle thread; therefore if we add $\Sigma I \div n = (\Sigma i \div n) \sec \delta$ to the average of the observed times of all complete transits, and reduce each thread of a broken transit separately by (102), all observations will be reduced to a common thread and we may proceed as before. It should be noted that the i's are positive for the threads on that side of the middle thread from which the star appears to come, and negative for the others.

These same equations worked backward may be used to determine the i's; thus $i = I \cos \delta$, and it is only necessary to observe the times of transit of a number of stars and then compute the values of the i's by the above formula. Stars near the pole should be selected, since with such stars I becomes large. It is better, however, if the transit has an eye-piece micrometer, to measure the intervals with the micrometer, and then compute i by the formula $i_n = (M_n — M_0)R$. Here M_n is the micrometer reading on any thread, M_0 is the same for the middle thread, and R is the value of one revolution of the micrometer screw. The method of finding R will be given in the next chapter.

CHAPTER XII.

THE ZENITH TELESCOPE

All the preceding methods for finding latitude depend upon the readings of a graduated circle. It is a finely graduated circle which reads to $10''$; a very fine one may read to $1''$, but in this case the errors of the divisions themselves may easily amount to more than this quantity. On the other hand it is an easy matter to make an eye-piece micrometer read directly to $0''.5$, and by estimation to $0''.05$; but an eye-piece micrometer can only measure a small arc, generally under $80'$. Can we devise a method of finding latitude which shall depend upon the measurement of a small arc? If so, we can use an eye-piece micrometer and greatly increase the accuracy without increasing the first cost of the instrument.

The method by which this is accomplished is as follows. Seleċt two stars, one crossing the meridian south and the other north of the zenith, and differing but little in their times of transit. From the south star we obtain $\phi = \delta_1 + Z_1$, and from the north star $\phi = \delta_2 - Z_2$. These two expressions for ϕ give at once the equation

$$\phi = \tfrac{1}{2}(\delta_1 + \delta_2) + \tfrac{1}{2}(Z_1 - Z_2)$$

In the above equation, $Z_1 - Z_2$ may be made as small as we wish by a proper selection of stars. To see how this principle can be used, let us suppose a theodolite provided with an eye-piece micrometer to be placed in the merid-

ian and pointed to a zenith distance equal to that of the average of the two stars. The first star will cross the field of the instrument either above or below its center. Let us bisect it with the micrometer wire as it crosses the meridian, the micrometer being so placed as to measure zenith distances. Read the micrometer and call the reading M_1. Now let us rotate the instrument 180° in azimuth leaving the telescope untouched; it will now point to the same zenith distance as before but on the opposite side of the zenith, and the second star will cross on the opposite side of the field of view from the first. Let us bisect it as before and call the micrometer reading M_2. The difference of the two micrometer readings, multiplied by the value of one revolution of the micrometer screw, gives the difference of the zenith distances of the two stars, and we will have

$$(103) \qquad \phi = \frac{1}{2}(\delta_1 + \delta_2) + \frac{1}{2}R(M_1 - M_2)$$

The above equation will only be true provided the inclination of the telescope was not changed during reversal. If, however, the level P and the circle $H_1 H_2$, Fig. X, page 60, be fastened to the telescope and turn with it, any change in the inclination of the telescope will produce an equal change in the level and a correction, in terms of the level readings, may be applied to formula (103). Again, the two stars are not at exactly the same zenith distance and will be differently refracted; hence we must apply to (103) a second correction due to refraction.

If, therefore, we give to a theodolite an eye-piece micrometer and so fasten the alidade level to the telescope that it may be clamped in any position, we will have an instrument especially adapted to the determination of latitude. Such an instrument is called a zenith telescope. As a matter of fact, in this instrument the telescope is gener-

ally fastened to the end of the horizontal axis rather than in the middle, and two adjustable stops are fastened to the horizontal circle for ease in reversal. From the above it is evident that the adjustments of a zenith telescope are the same as for a theodolite. It can be placed in the meridian like a transit or, if it is provided with a horizontal circle, by pointing on Polaris, noting the corresponding time, and computing the azimuth of Polaris, from which we can compute the two meridian settings and set the stops accordingly.

Level Correction

In deriving the formula for the level correction, we shall suppose that the level is graduated continuously from one end. Let B be the angle between the plumb line and the line joining the center of the objective and the micrometer wire when the micrometer reads M_0 and the level reads L, and B_0 the value of this angle when the level reads L_0. Now suppose the zero of the level to be at the south end and the telescope, carrying the level with it, to be tilted up towards the zenith. The bubble will take a new position where the reading of the center of the bubble will be $\frac{1}{2}(Ls_1 + Ls_2)$, Ls_1 and Ls_2 being the readings of the two ends. This quantity will evidently be greater than L_0 if the objective is north of the zenith, and less than L_0 if the objective is south of it. In either case the angle B will be less than B_0, and we will have $B = B_0 \pm [\frac{1}{2}(Ls_1 + Ls_2) - L_0]$, where the upper sign is to be used for stars south of the zenith. Now suppose the instrument rotated 180° in azimuth and set for the second star. The zero of the level will then be at the north end of the scale, and B_0 and L_0 will remain as before. If we now tilt the objective towards the zenith the bubble will take the new position $\frac{1}{2}(Ln_1 + Ln_2)$, and

this will be greater than L_0 for stars south of the zenith, and less than L_0 for stars north of the zenith, but in either case B will be diminished and we will have as before $B = B_0 \mp [\frac{1}{2}(Ln_1 + Ln_2) - L_0]$, the upper sign being used for south stars.

The true zenith distance of a star will equal this angle B plus a term M, depending upon the micrometer reading. Let Zn and Zs be the true zenith distances of the two stars; and suppose that we start with the north star, and with the zero of the level on the south. We will then have

$$Zn = B_0 - [\frac{1}{2}(Ls_1 + Ls_2) - L_0] + M_1$$

For the second star the zero of the level will be on the north, the star will be south of the zenith, and we have

$$Zs = B_0 - [\frac{1}{2}(Ln_1 + Ln_2) - L_0] = M_2$$

From these two equations we find

$$\frac{1}{2}(Zs - Zn) = \frac{1}{2}(M_2 - M_1) +$$
$$\frac{1}{4}[Ls_1 + Ls_2 - (Ln_1 + Ln_2)]$$

No matter in what position we start the instrument, we find that the last term comes out the same. This is the level correction and may be expressed in the form of a rule as follows: from the sum of the level readings, zero on the south, subtract the sum of the level readings, zero on the north, and divide the result by four.

Level: Zero at One End

$$\frac{1}{4}D[Ls_1 + Ls_2 - (Ln_1 + Ln_2)]$$

Here s and n denote the position of the zero of the level.

Level: Zero at the Middle

$$\frac{1}{4}D[Ln_1 + Ln_2 - (Ls_1 + Ls_2)]$$

Here s and n denote the ends of the bubble.

Refraction Correction

In Chapter III we saw that the refraction could be closely represented by the formula $r = 57''.7 \, tg \, Z$. If we

differentiate the above we will have dr $= 57''.7 \sec^2 Z\ dZ$, where dZ must be expressed in radians. In the last equation dZ may be taken as $R(M_1 - M_2)\sin 1'$, since the other corrections are all small. Expressing $R(M_1 - M_2)$ in minutes and calling Δr the refraction correction, we will have, since only half the difference of the zenith distances enters into formula (103),

$$\Delta r = 28''.85 \sin 1' \sec^2 Z \times R(M_1 - M_2) =$$
$$\tfrac{1}{2}\Delta' r\ R(M_1 - M_2)$$

Table VIII gives the values of $\Delta' r$ for the argument Z.

It sometimes happens that the observer is unable to bisect the star exactly as it crosses the meridian; in this case a correction, $[6.1347]t^2 \sin 2\delta$,[1] should be applied to the computed latitude. In this expression t is the star's hour angle in seconds of time, and the quantity within the brackets is a logarithm. This quantity is called the reduction to the meridian. Collecting these results, formula (103) becomes

(104) $\phi = \tfrac{1}{2}(\delta_1 + \delta_2) + \tfrac{1}{2}R(M_1 - M_2) +$
            level correction + refraction correction +
                    reduction to the meridian

Determination of the values of R and D

The best method of finding the value of one revolution of the micrometer screw, (R), is by observing the transits of a close circum-polar star over the micrometer thread when the star is near elongation, setting the micrometer ahead one revolution between successive transits. A star is said to be at elongation when its azimuth from the north is a maximum. A glance at Fig. I, page 6, will show that this must occur when the angle PSZ $= 90°$. Applying the fundamental formulæ of Chapter II to this triangle, we will have, after reduction,

1 See Chauvenet's "Spherical and Practical Astronomy" vol. II, page 347.

$$\cos Z_0 = \mathrm{cosec}\ \delta \sin \phi \qquad \cos t_0 = \mathrm{tg}\ \phi \cot \delta$$

Let T_0 be the chronometer time of elongation and we have

$$T_0 = a + t_0 - \Delta T$$

Some twenty or thirty minutes before this computed time, set the telescope at the computed zenith distance and revolve it in azimuth until the star appears in the field of view. Set the micrometer wire at some even revolution, slightly in advance of the star, and note the time of transit; then set it ahead one revolution and note the time of the second transit. Keep this up until the star has crossed the field of view, reading the level frequently.

Let Z be the zenith distance of the star when it is bisected by the cross-wire at any given setting of the micrometer and t the corresponding hour angle. Then

(105) $\qquad \cos Z = \sin \phi \sin \delta + \cos \phi \cos \delta \cos t$

The above equation may be written

$$\cos[Z_0 + (Z - Z_0)] = \sin \phi \sin \delta +$$
$$\cos \phi \cos \delta \cos[t_0 + (t - t_0)]$$

Expanding the above we have

$$\cos Z_0 \cos(Z - Z_0) - \sin Z_0 \sin(Z - Z_0) =$$
$$\sin \phi \sin \delta + \cos \phi \cos \delta \cos t_0 \cos(t - t_0) -$$
$$\cos \phi \cos \delta \sin t_0 \sin(t - t_0)$$

In the above equation $Z - Z_0$ is always a small quantity and we may replace its sine by its arc and its cosine by unity. The arc $t - t_0$ is not as small as $Z - Z_0$ but, as $\cos \phi \cos \delta \cos t_0 \cos(t - t_0)$ is a small term, since δ is nearly 90°, in it we may replace $\cos(t - t_0)$ by unity without appreciable error. We will then have

(106) $\qquad \cos Z_0 - (Z - Z_0) \sin Z_0 = \sin \phi \sin \delta +$
$$\cos \phi \cos \delta \cos t_0 - \cos \phi \cos \delta \sin t_0 \sin(t - t_0)$$

Formula (105) gives

$$\cos Z_0 = \sin \phi \sin \delta + \cos \phi \cos \delta \cos t_0$$

Combining this with (106) we have, after reduction,

(107) $(Z - Z_0)\sin Z_0 = \cos \phi \cos \delta \sin t_0 \sin(t - t_0)$

If we apply formula (1), page 10, to the triangle SPZ, Fig. I, page 6, we will have, when $Z = Z_0$ and hence the angle PSZ $= 90°$,

$$\sin Z_0 = \sin t_0 \cos \phi$$

Combining this with (107) we have

$$Z - Z_0 = \cos \delta \sin(t - t_0)$$

Obviously $t - t_0 = T - T_0$, and if we let $Z - Z_0 = A$, we will have, after dividing by sin 1″,

(108) $$Z - Z_0 = A = \sin(T - T_0)\frac{\cos \delta}{\sin 1''}$$

If we then compute the values of A_1, A_2, A_3, . . . A_n for each observed transit by formula (108), we will have

$$R = 0.1(A_{11} - A_1) = 0.1(A_{12} - A_2) = \text{etc.}$$

If the level has changed during the observations by an amount ΔL, we must correct the angles $A_{11} - A_1$, etc., by $\Delta L \times D$, D being the value of one division of the level. This correction is negative when the objective moves with the star and positive when it moves against it.

The value of R thus determined is affected by refraction, and the true value, R_0, is given by the equation

(109) $$R_0 = R - \Delta'r \times R'$$

The value of $\Delta'r$ is given in Table VII.

The above method may be used to determine the value of one revolution of the eye-piece micrometer of a transit instrument. In this case T_0 becomes the chronometer time of culmination, i. e., $a - \Delta T$, and $Z - Z_0$ the angle subtended at the objective between the axis of collimation and the micrometer thread at a given setting.

The best method of finding D is to point the telescope at a distant terrestrial mark, run the bubble to one end of the level, and then bisect the mark with the micrometer wire; then run the bubble to the other end of the level

and bisect the mark a second time. Evidently, then, we will have measured the same angle with both the level and the micrometer, whence it follows that

$$(110) \qquad D = \frac{M_1 - M_2}{N} R$$

In this equation M_1 and M_2 are the micrometer readings in the two positions and N is the number of divisions moved over by the bubble. This equation contains R while the equation for R contains D, but no appreciable error will be introduced if we use a value of R in (110) obtained by neglecting the level correction.

The Observing List

Before beginning work with the zenith telescope the student should prepare a list of all stars suitable for this purpose. The conditions to be met are as follows:—

1. The difference of their zenith distances must be less than the field of view of the instrument, which can be easily measured by running the micrometer first to one edge of the field and then to the other.

2. The stars should not cross the meridian too close together, for then the observer will not have time to reverse his instrument. The limit is about two minutes.

3. The stars should not cross the meridian too far apart, for then the instrument might change between the observations and furthermore a long night's work would show but few observations. The limit is about twenty minutes.

4. The stars must be sufficiently bright to be seen. For most field instruments the sixth magnitude is about the faintest star that can be observed.

5. The stars should not be at too great a zenith distance; 30° to 40° is about a maximum.

The number of stars given in the Ephemerides is by

no means large enough to furnish a sufficient number of pairs, and the student must select them from the star catalogues. One of the best for this purpose is the Greenwich Ten Year Catalogue for 1880. Here the north polar distances are given rather than the declinations; and we have as the condition to be met by the stars forming a pair, $p = 180° — 2\phi — p' \pm$ field. Write the value of $180° — 2\phi$ on a slip of paper, and, turning to the point in the catalogue at which we expect to begin work, subtract the north polar distance of the first star from $180° — 2\phi$; then run down the catalogue for the next twenty or twenty-five minutes of right ascension, marking all stars which satisfy condition (1). Take the next star and proceed in a similar manner until three or four hours of right ascension have been covered. It will be found that many of the pairs will overlap, so that if one pair is observed others can not be. Select from this list of all possible pairs ten or a dozen of the best, i. e., stars neither too bright nor too faint, too near together nor too far apart in right ascension, but above all stars whose proper motions are given. For this reduced list compute the mean a_0 and δ_0 by the methods of Chapter V, δ_0 being computed rigorously and a_0 by neglecting the secular variation and the proper motion.

We are now ready to prepare the observing list. This list should state the number or name of the star, its mean right ascension, whether it is north or south of the zenith, the setting, i. e., the average zenith distance of the pair, and lastly the approximate micrometer reading. This last is very important, as it enables the observer to know about where in the field of view to look for the star. None of the above needs further explanation but the last. For this purpose it is necessary to know how the microm-

eter increases or decreases with the zenith distance. To determine this, point the telescope at some distant mark and with the micrometer wire set at the middle of the field bisect the top of the mark. Then run the micrometer wire towards the bottom of the mark and note whether its reading has increased or diminished. Obviously, in passing from the top to the bottom of the mark we have increased the zenith distance and, if at the same time the micrometer has increased, the two go together; if not, they go opposite. The observer should then make out a table similar to the following:—

$$\text{Objective}\begin{Bmatrix} \text{south} \\ \text{south} \\ \text{north} \\ \text{north} \end{Bmatrix} \text{Telescope}\begin{Bmatrix} \text{east} \\ \text{west} \\ \text{east} \\ \text{west} \end{Bmatrix} \begin{matrix} \text{Micrometer increases} \\ \text{as the} \\ \text{Zenith Distance} \end{matrix} \begin{Bmatrix} \text{decreases} \\ \text{increases} \\ \text{increases} \\ \text{decreases} \end{Bmatrix}$$

The distance at which any star will cross the meridian from the middle of the field is equal to the difference between its zenith distance and the setting of the instrument. If we call this distance $S — Z$, and M and M_0 the micrometer settings for the star and for mid-field respectively, we will have

$$M = M_0 \pm (S — Z) \div R$$

Here the sign is to be determined by the above table.

As an example we may take the following pair:

7 Trianguli $a = 2^h \, 09^m$ $\delta = + 32° \, 53'$
62 Andromedæ $a = 2^h \, 13^m$ $\delta = + 46° \, 53'$

$\phi = 40° \, 00'$, setting $= 7° \, 00'$, $R = 0'.80$, and $M_0 = 27$.

For the first star the zenith distance is $7° \, 07'$ and hence $S — Z = 7'$, and $7' \div 0'.80 = 9$ revolutions. Suppose the telescope is west, the objective will be south for the first star, and the table shows that in this position of the instrument, micrometer and zenith distance go together. The zenith distance of the first star is greater than the setting and hence $M = 27 + 9 = 36$. In the

same way we find M = 27 — 9 = 18 for the first star when the telescope is east.

Process of Reduction

In reducing observations made with the zenith telescope, it is only necessary to compute the reductions to the apparent places by the methods of Chapter V, add these to the mean declinations, and then compute the latitude by formula (104). It should be added that the micrometer correction is generally so large that its sign is self-evident; in cases of doubt, however, the sign may be determined from the table on the preceding page. The refraction correction always has the same sign as the micrometer correction.

Example

Observing List and Record of Observations

Star No.	Mag.	Right Ascension	N or S	Setting	Telescope E	Telescope W	Micrometer Reading	Level N	Level S
		h. m. s.		o ′				d.	d.
2911	2.3	18 12 37	N	3 03	11. 89		87.793	66.0	30.0
2923	4.4	18 16 28	S				9.188	31.0	67.3
2937	3.9	18 19 23	S	18 30	60 40		38.170	67.5	31.0
2952	4.8	18 22 30	N				59.070	31.8	68.3

Reduction

Star No.	Apparent Declination	Average Declination	Microm. Correction	Ref.	Level	Seconds of Latitude
	o ′ ″	o ′ ″	′ ″	″	″	″
2911	42 07 57.8	39 04 47.6	55 03.0	0.9	— 1.2	50.3
2923	36 01 37.4					
2937	21 43 51.9	40 14 28.4	14 38.2	0.3	— 0.8	49.1
2952	58 45 04.8					

DATE:— August 17th, 1903. R = 84″.040. D = 2″.o. φ = 39° 59′ 49″.7. M$_0$ = 50. For the micrometer settings, use the table on page 135.

TABLES

Refraction Tables

TABLE I. MEAN REFRACTION

Apparent Altitude °	Mean Refraction ″	Apparent Altitude °	Mean Refraction ″
10.0	319	30	101
10.5	305	31	97
11.0	291	32	93
11.5	279	33	89
12.0	268	34	86
12.5	257	35	83
13.0	247	36	80
13.5	238	37	77
14.0	230	38	74
14.5	222	39	72
15.0	214	40	69
15.5	207	41	67
16.0	200	42	65
16.5	194	43	62
17.0	188	44	60
17.5	183	45	58
18.0	177	46	56
18.5	172	47	54
19.0	168	48	52
19.5	163	49	51
20.0	159	50	49
21.0	151	55	41
22.0	143	60	34
23.0	136	65	27
24.0	130	70	21
25.0	124	75	16
26.0	119	80	10
27.0	114	85	05
28.0	109	90	00
29.0	105		

TABLE II. LOG $\psi_1(T)$

Air Temp. Degrees F. °	Logarithm Air Temp. Factor ″
6	0.0390
9	0.0363
12	0.0335
15	0.0308
18	0.0280
21	0.0253
24	0.0226
27	0.0200
30	0.0173
33	0.0147
36	0.0120
39	0.0094
42	0.0068
45	0.0043
48	0.0017
51	9.9992
54	9.9966
57	9.9941
60	9.9916
63	9.9891
66	9.9866
69	9.9842
72	9.9817
75	9.9793
78	9.9769
81	9.9745
84	9.9721
87	9.9697
90	9.9673
93	9.9650
96	9.9626

TABLE III. LOG $\psi_2(B)$

Barometer in Inches	Logarithm Barometer Factor ″
27.8	9.9669
27.9	9.9685
28.0	9.9700
28.1	9.9716
28.2	9.9731
28.3	9.9747
28.4	9.9762
28.5	9.9777
28.6	9.9792
28.7	9.9808
28.8	9.9823
28.9	9.9838
29.0	9.9853
29.1	9.9868
29.2	9.9883
29.3	9.9897
29.4	9.9912
29.5	9.9927
29.6	9.9942
29.7	9.9956
29.8	9.9971
29.9	9.9986
30.0	0.0000
30.1	0.0014
30.2	0.0029
30.3	0.0043
30.4	0.0057
30.5	0.0072
30.6	0.0086
30.7	0.0100
30.8	0.0114

General Tables

TABLE IV. LOG $\psi_3(t)$		TABLE VII. LOG k		TABLE VIII. $\Delta'r$	
Attached Thermom. Degrees F.	Logarithm Att. Ther. Factor	Daily Rate	Logarithm of k	Zenith Distance	$\dfrac{dr}{dz}$
o		s		o	"
0	0.0020	— 30	9.99970	0	0.0168
10	0.0016	— 20	9.99980	5	0.0169
20	0.0012	— 10	9.99990	10	0.0173
30	0.0009	± 00	0.00000	15	0.0180
40	0.0005	+ 10	0.00010	20	0.0190
50	0.0000	+ 20	0.00020	25	0.0205
60	9.9996	+ 30	0.00030	30	0.0225
70	9.9992			35	0.0250
80	9.9989			40	0.0286
90	9.9985			45	0.0336
100	9.9981				

TABLE VI. $n = \dfrac{2\sin^4\frac{1}{2}t}{\sin 1''}$						TABLE IX. $\dfrac{p}{8''.8\cos h}$	
t	n	t	n	t	n	h	p
m	"	m	"	m	"	o	"
10	0.1	17	0.8	24	3.1	20	8.3
11	0.1	18	1.0	25	3.6	30	7.6
12	0.2	19	1.2	26	4.3	40	6.7
13	0.3	20	1.5	27	5.0	50	5.7
14	0.4	21	1.8	28	5.7	60	4.4
15	0.5	22	2.2	29	6.6	70	3.0
16	0.6	23	2.6	30	7.6	80	1.5

Table V.

$$m'' = \frac{2 \sin^2 \tfrac{1}{2}t}{\sin 1''}$$

t	0^s	10^s	20^s	30^s	40^s	50^s
0^m	0.0″	0.0″	0.2″	0.5″	0.9″	1.4″
1	2.0	2.7	3.5	4.4	5.5	6.6
2	7.9	9.2	10.7	12.3	14.0	15.8
3	17.7	19.7	21.8	24.0	26.4	28.9
4	31.4	34.1	36.9	39.8	42.8	45.9
5	49.1	52.4	55.8	59.4	63.0	66.8
6	70.7	74.7	78.8	83.0	87.3	91.7
7	96.2	100.8	105.6	110.4	115.4	120.5
8	125.7	130.9	136.3	141.8	147.5	153.2
9	159.0	165.0	171.0	177.2	183.5	189.8
10	196.3	202.9	209.6	216.4	223.4	230.4
11	237.5	244.8	252.2	259.6	267.2	274.9
12	282.7	290.6	298.6	306.7	315.0	323.3
13	331.7	340.3	349.0	357.7	366.6	375.6
14	384.7	393.9	403.3	412.7	422.2	431.9
15	441.6	451.5	461.5	471.6	481.7	492.0
16	502.5	513.0	523.6	534.3	545.2	556.1
17	567.2	578.4	589.6	601.0	612.5	624.1
18	635.9	647.7	659.6	671.6	683.8	696.0
19	708.4	720.9	733.5	746.2	759.0	771.9
20	784.9	798.0	811.3	824.6	838.0	851.6
21	865.3	879.0	893.0	907.0	921.1	935.2
22	949.6	963.9	978.5	993.2	1008.0	1022.8
23	1037.8	1052.8	1068.1	1083.3	1098.8	1114.3
24	1129.9	1145.6	1161.5	1177.5	1193.5	1209.6
25	1225.9	1242.3	1258.8	1275.4	1292.2	1309.0
26	1325.9	1342.9	1360.1	1377.3	1394.7	1412.2
27	1429.7	1447.4	1465.2	1483.1	1501.1	1519.2
28	1537.5	1555.8	1574.3	1592.7	1611.5	1630.2
29	1649.0	1668.0	1687.2	1706.3	1725.6	1745.1

Logarithms of Numbers

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
N	0	1	2	3	4	5	6	7	8	9

Logarithms of Numbers

N	0	1	2	3	4	5	6	7	8	9
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
N	0	1	2	3	4	5	6	7	8	9

Logarithms of Numbers

N	0	1	2	3	4	5	6	7	8	9
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9